Chapter 3

Number System

1. **Introduction Principle :** Let $\{T(n): n \in N\}$ be a set of statements, one for each natural number n. If (i), $T(a)$ is true for some $a \in N$ and (ii) $T(k)$ is true implies $T(k+1)$ is true for all $k \geq a$, then $T(m)$ is true for all $n \geq a$.

2. **The greatest integer function :** [ ] is defined by selling $[x] =$ the greatest integer not exceeding x, for every real x.

3. If a/b and a/c, then $a/pb+qc$ (linearity property).

4. **Euclid's Algorithm :** The a and b be two non-zero integers. Then (a, b) [gcd of (a, b)] exists and is unique. Also, there exists integers m and n method $(a, b) = am + bn$.

5. **Congruencies :** Let a, b in be integers, $m > 0$. Then we say that a is congruent to b modulo m if, $m/(a-b)$. We denote this by $a \equiv b(\text{mod}\, m)$.

6. Let $a \equiv b(\text{mod}\, m)$ and $c = d$ (mod m) Then

   (i) $a + c \equiv b + d(\text{mod}\, m)$

   (ii) $a - c \equiv b - d(\text{mod}\, m)$

   (iii) $ac \equiv bd(\text{mod}\, m)$

   (iv) $pa + qc \equiv pb + qd(\text{mod}\, m)$ for all integers p and q.

   (v) $a^n \equiv b^m$ mod m for all positive integer m.

   (vi) $f(a) \equiv f(b)(\text{mod}\, m)$ for every polynomial with integer coefficients.

7. An integer x_0 satisfying the linear congruence

   $ax \equiv b(\text{mod}\, m)$ has a solution. Further more,

   if x_0 is a solution, then the set of all solutions is precisely $(x_0 + km) : (\alpha \in Z)$.

8. Let N be a positive integer greater than 1, say $N = a^p b^q c^r ....$ where $a, b, c...$ are distinct (different primes and $p, q, r...$ are positive integers. The number of ways in which N can be resolved into two factors is

$$\frac{1}{2}(p+1)(q+1)(r+1)...$$

9. Number of ways in which a composite number can be resolved into two factors, which are prime to each other, is 2^{n-1} where n is the number of distinct prime factors in the expression for N.

10. Let N be a positive integer quarter than 1 and let $N = a^{pq}b^{q}c^{r}...$ where $a, b, c...$ are distinct primes and $p, q, r...$ positive integers. then the sum of all the divisors in the product is equal to

$$\frac{a^{p+1}-1}{a-1}.\frac{b^{q+1}-1}{b-1}.\frac{c^{r+1}-1}{c-1}$$

11. the highest power of prime p which is contained in $n!$ is equal to

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+...$$

where [ ] is the greatest integer function.

12. **Euler's Totient Function :** Let N be any positive integer $>1.$ the number of all positve integers less than N and prime to it is denoted by $\phi(N)$. It is obvious $\phi(2)=1, \phi(3)=2,$ $\phi(4)=2, \phi(5)=4, \phi(6)=2...$ The function ϕ is called **Euler's Totient function**.

If $a, b, ...$ are prime to each other, then

$$\phi(ab) = \phi(a).\phi(b)$$

or $$\phi(abcd\ ...) = \phi(a)\phi(b)\phi(c)...$$

If $$N = a^{p}b^{q}c^{r}...$$

where a, b, c are distinct primes and p, q, r are positive integers, then

$$\phi(N) = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)...$$

Euler's Theorem : If x be any positive integer prime to N.

Then $$x^{\phi(N)} \equiv 1(\text{mod } N)$$

13. **Fermat's Little Theorem :** If p is a prime and n is prime to p then $n^{p-1} = 1(\text{mod } p)$.

14. **Wilson's Theorem :** If p is a prime, their

$$(p-1)! \equiv 0(\text{mod } p)$$

Conversely, if $(n-1)!+1 \equiv 0\ (\text{mod } n),$ their n is a prime.

Example : Calculate $5^{2039}(\text{mod } 41)$.

Solution : Since 41 is prime and $(5, 41) = 1,$ therefore by Fermat's

$5^{40} \equiv 1(\text{mod } 41)$

by division algorithm, $2039 = (50 \times 40) + 39$

$\therefore \quad 5^{2039} = 5^{50.40+39} = (5^{40})^{50}5^{39} = 1^{50}.5^{39}(\text{mod } 41) = 5^{39}(\text{mod } 41)$

To calculate $5^{39} \pmod{41}$, we first calculate $5^n \pmod{41}$ where $n = 2, 4, 8, 16, 32$.

$$5^2 = 25 \equiv -16 \pmod{41}$$

$$5^4 = 216 \equiv 10 \pmod{41}$$

$$5^8 = 100 \equiv 18 \pmod{41}$$

$$5^{16} \equiv 324 \equiv -4 \pmod{41}$$

$$5^{32} \equiv 16 \pmod{41}$$

$$\therefore \quad 5^{39} = 5^{32}.5^4.5^2.5 \equiv 16.10.(-16).5$$

$$\equiv 33 \pmod{41}$$

$$\therefore \quad 5^{2039} \pmod{41} = 33.$$

Question 1. Find the largest positive integer n such that $n^3 + 100$ is divisible by $(n+10)$.

Solution : Using modulo $(n+10)$ numbers, we see that

$$n + 10 \equiv 0[\bmod (n+10)]$$

i.e.,

$$n = -10[\bmod(n+10)]$$

$$n^3 \equiv (-10)^3[\bmod(n+10)]$$

$$= -1000([\bmod(n+10)]$$

$$\therefore \quad n^3 + 100 \equiv (-1000+100)[\bmod(n+10)]$$

$$\equiv -900[\bmod(n+10)]$$

Now, we want $(n+10)$ to divide $n^3 + 100$, implying that $(n+10)$ should divide -900.

The largest such n is $900 - 10 = 890$, as $(n+10)$ cannot be greater than $|-900| = 900$ and the greatest divisor of $|-900|$ is 900.

so the largest positive integer n, such that $n^3 + 100$ is divisible by $(n+10)$ is $n = 890$.

Note : $900 = 3^2 \times 2^2 \times 5^2$ has 27 divisors an each divisor greater than 10, gives a corresponding value for n they are 2, 5, 8, 10, 15, 20, 26, 35, 40, 50, 65, 80, 90, 140, 170, 215, 290, 440 and 890.

Question 2. When the numbers 19779 and 17997 are divided by a certain three digit number, they leave the same remainder. Find this largest such divisor and the remainder. How many such divisors are there ?

Solution : Let the divisor be d an the remainder be r.

The by Euclidean Algorithm, we find

$$19779 = dq_1 + r \qquad \ldots(1)$$

and

$$17997 = dq_2 + r \qquad \ldots(2)$$

by subtracting Eq. (2) from Eq. (1), we get

$$1782 = d(q_1 - q_2)$$

$\therefore$ d is a three digit divisor of 1782.

therefore, possible values of d are 891, 594, 297, 198.

Hence largest three digit divisor is 891 and the remainder is 177.

Question 3. Find the number of all rational numbers $\frac{m}{n}$ such hat (i) $0 < \frac{m}{n} < 1$, (ii) m and n are relatively prime and (iii) $m.n = 25!$.

Solution : It is given that

$$m \times n = 25! = 2^{22} \times 3^{10} \times 5^6 \times 7^1 \times 11^1 \times 13^1 \times 17^1 \times 19^1 \times 23^1$$

Thus 25! is the product of powers of 9 prime numbers.

The number of ways in which 25! can be written as the product of two relatively prime numbers m and n is 2^9, which leads to 2^9 factors, exactly half of which, are such that $\frac{m}{n}$ is less than 1. There are 2^8 such fractions.

Question 4. Determine all positive integers n for which $2^n + 1$ is divisible by 3.

Solution : $2^n + 1 = 2^n + 1^n$

If n is odd, then $(2+1)$ is a factor. Thus for all odd values of n, $2^n + 1$ is divisible by 3.

Aliter :

$$2^1 \equiv 2 (\text{mod } 3)$$

$$2^2 \equiv 1 (\text{mod } 3)$$

$$2^3 \equiv 2 (\text{mod } 3) \text{ and so on}$$

$$2^{2m+1} \equiv 2 (\text{mod } 3) \text{ and } 2^{2m} \equiv (2^2)^m = 4^m \equiv 1 \text{ mod } (3)$$

$$2^n + 1 = 2 + 1 = 0 \text{ mod } 3 \text{ if } n \text{ is odd.}$$

and $2^n + 1 = 1 + 1 = 2 \text{ mod } 3$ if n is even.

$\therefore$ $2^n + 1$ is divisibly by 3 if n is an odd number

Question 5. Prove that $[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345$ has no solution.

Solution :

$$12345 \le x + 2x + 4x + 8x + 16x + 32x = 63x$$

$$\therefore \quad x \ge \frac{12345}{63} = 195\frac{20}{21}$$

When $x = 196$, the L.H.S. of the given equation becomes 12348.

$$\therefore \quad 195\frac{20}{21} \le x < 196.$$

Consider x in the interval $\left(195\frac{31}{32}, 196\right)$. the L.H.S. expression of the given equation

$$= 195 + 0 + 390 + 1 + 780 + 3 + 1560 + 7 + 3120 + 15 + 6240 + 31$$
$$= 12342 < 12345$$

When $x < 195\frac{31}{32}$, the L.H.S. is less than 12342.

$\therefore$ for no value of x, The given equality will be satisfied.

Question 6. Three consecutive positive integers raised to the first, second and third powers respectively, when added, make a perfect square, the square root of which is equal to the sum of the three consecutive integers. Find these integers.

Solution : Let $(n-1)$, n, $n+1$ be the three consecutive integers.

Then $(n-1)^1 + n^2 + (n+1)^3 = (3n)^2 = 9n^2$

$\Rightarrow \quad n - 1 + n^2 + n^3 + 3n^2 + 3n + 1 = 9n^2$

$\Rightarrow \quad n^3 - 5n^2 + 4n = 0$

$\Rightarrow \quad n(n-1)(n-4) = 0$

$\Rightarrow \quad n = 0$ or $n = 1$ or $n = 4$,

but $n = 0$ and $n = 1$ will make the consecutive integers $-1, 0, 1$ and 0, 1, and 2, which contradicts the hypothesis that the consecutive integers are all greater than zero.

Hence $n = 4$, corresponding to which the consecutive integers are 3, 4 and 5.

Question 7. Show that $1^{1997} + 2^{1997} + ... + 1996^{1997}$ is divisible by 1997.

Solution : We shall make groups of the terms of the expression as follows :

$$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + ... + (998^{1997} + 999^{1997})$$

Here each bracket is of the form $(a_i^{2n+1} + b_i^{2n+1})$ is divisible by $(a_i + b_i)$.

But $(a_i + b_i) = 1997$ for all i.

$\therefore$ Each bracket and hence, their sum is divisible by 1997.

Question 8. A four digit number has the following properties :

(a) It is a perfect square

(b) The first two digits are equal

(c) The last two digits are equal.

Find all such numbers.

Solution : Let $N = aabb$ be the representation of such a number.

$$1 \le a \le 9, \; 0 \le b \le 9$$

Then $N = 1000a + 100a + 10b + b = 1100a + 11b = 11(100a + b)$

Since N is a perfect square, and 11 is a factor of N

$aob = 100a + b$ must be a multiple of 11, *i.e.*, aob should be divisible by 11 where a, o and b are digits of $100a + b$ and hence $b = 11 - a$.

The last 2 digits of a perfect square where both the digits are equal is only 44.

So $\quad b = 4$

$\therefore \quad a = 7$

$N = 7744$ is the only possibility

$n = 11 \times 704 = 11 \times 11 \times 64 = 88^2.$

$\therefore$ This is the only solution.

Question 9. Show that $2^{55} + 1$ is divisible by 11.

Solution :

$$2^5 = 32 = (-1) \pmod{11}$$

$$255 = (2^5)^{11} = (-1)^{11} \pmod{11}$$

$$= -1 \pmod{11}$$

So $\quad 2^{55} + 1 = 0 \pmod{11}$

$\therefore$ It is a multiple of 11.

Question 10. The equation $x^2 + px + 9 = 0$ has rational roots, where p and q are integers. Prove that the roots are integers.

Solution : $x = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$,

since the roots are rational, $p^2 - 4q$ is a perfect square.

If p is even, p^2 and $4q$ are even and hence $p^2 - 4q$, is an even integer and hence, $-p \pm \sqrt{p^2 - 4q}$ is an even integer and hence, $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ is an integer.

If p is odd, $(p^2 - 4q)$ is odd and $-p \pm \sqrt{p^2 - 4q}$ is an even integer and hence, $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ is an integer and hence, the result.

Question 11. Find all pairs of natural numbers, the difference of whose squares is 45.

Solution : Let x and y be the natural numbers such that $x^2 - y^2 = 45$ where $x > y$.

$\Rightarrow \quad (x - y)(x + y) = 45$

so, both $(x - y)$ and $(x + y)$ are the divisors of 45 and $x + y > x - y$, where x and y are positive integers.

So, $\quad x - y = 1, \quad x + y = 45 \quad \ldots(1)$

$$x - y = 3, \qquad x + y = 15 \qquad \ldots(2)$$

$$x - y = 5, \qquad x + y = 9 \qquad \ldots(3)$$

Solving (1), (2) and (3), we get

(a) $x = 23,\ y = 22$

(b) $x = 9,\ y = 6$

(c) $x = 7,\ y = 2$

so, the pairs of numbers satisfying the condition are (23, 22), (9, 6), (7, 2).

Question 12. Show that any circle with centre $(\sqrt{2}, \sqrt{3})$ cannot pass through more than one lattice point.

[Lattice points are points in cartesian plane, whose abscissa and ordinate both are integers.]

Solution : If possible, let $(a, b), (c, d)$ be two lattice points on the circle with $(\sqrt{2}, \sqrt{3})$ as centre and radius 'r'.

$$(a - \sqrt{2})^2 + (b - \sqrt{3})^2 = r^2 = (c - \sqrt{2})^2 + (d - \sqrt{3})^2$$

$$\Rightarrow \quad a^2 + b^2 - c^2 - d^2 = 2(\sqrt{2}a + \sqrt{3}b) - 2(\sqrt{2}c + \sqrt{3}d)$$

$$= 2\sqrt{2}(a - c) + 2\sqrt{3}(b - d)$$

Since $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers and $2, a, b, c$ and d are integers $2(a - c)\sqrt{2} + 2(b - d)\sqrt{3}$ is irrational as $\sqrt{2}$ and $\sqrt{3}$ are unlike irrational numbers and hence, adding $k\sqrt{2}$ and $l\sqrt{3}$ where k and l are integers does not give a rational number.

But the left hand side $a^2 + b^2 - c^2 - d^2$ is an integer. It is a contradiction and thus, the circle with $(\sqrt{2}, \sqrt{3})$ as centre can pass through utmost one lattice point.

[**Note :** You may know that the equation of the circle whose centre is $(-g, -f)$ and radius r given by

$$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 - r^2 = 0$$

for the circle $$x^2 + y^2 - 2\sqrt{2}x - 2\sqrt{3}y + (5 - r^2) = 0.$$

If it passes through the origin, then

$5 - r^2 = 0$ or $r = \sqrt{5}$, so the one lattice point that lies on the circle with centre $(\sqrt{2}, \sqrt{3})$ and radius $\sqrt{5}$ is (0, 0). But it is not necessary that there exists at least one lattice point for circles with such centres.]

Question 13. Find all positive integers n for which $n^2 + 96$ is perfect square.

Solution : Let $n^2 + 96 = k^2$, where $k \in N$.

Then $k^2 - n^2 = 96$

$$(k - n)(k + n) = 96 = 3^1 \times 2^5$$

Clearly $k > n$ and hence, $k + n > k - n > 0$.

Since 3 is the only odd factor, both k and n are integers. We must have $k + n$ and $k - n$ both to be either even or odd. (If one is odd and the other even, then k and n do not have integer solutions). Also both $k + n$ and $k - n$ cannot be odd as the product is given to be even. So the different possibilities for $k + n$, $k - n$ are as follows.

$$k - n = 2 \qquad k + n = 48 \qquad \ldots(1)$$
$$k - n = 4 \qquad k + n = 24 \qquad \ldots(2)$$
$$k - n = 6 \qquad k + n = 16 \qquad \ldots(3)$$
$$k - n = 8 \qquad k + n = 12 \qquad \ldots(4)$$

So, solving separately Eqns. (1), (2), (3) and (4), we get $n = 23,\ 10, 5, 2$.

So, there are exactly four values for which $n^2 + 96$ is a perfect square.

$$n = 23 \text{ gives } 23^2 + 96 = 625 = 25^2.$$
$$n = 10 \text{ gives } 10^2 + 96 = 196 = 14^2.$$
$$n = 5 \text{ gives } 5^2 + 96 = 121 = 11^2.$$
$$n = 2 \text{ gives } 2^2 + 96 = 100 = 10^2.$$

Question 14. There are n necklaces such that the first necklace contains 5 beads, the second contains 7 beads and in general, the tth necklace contains i beads more than the number of beads in $(i - 1)$ th necklace. Find the total number of beads in all the n necklaces.

Solution : Let us write the sequence of the number of beads in the 1st, 2nd, 3rd, ..., nth necklaces.

$$= 5,\ 7,\ 10,\ 14,\ 19, ...$$

$$= (4 + 1),\ (4 + 3),\ (4 + 6),\ (4 + 10),\ (4 + 15),\ ...,\ \left[4 + \frac{n(n+1)}{2}\right]$$

$$S_n = \text{Total number of beads in the } n \text{ necklaces}$$

$$S_n = \Big(\underbrace{4 + 4 + ... + 4}_{n \text{ times}}\Big) + 1 + 3 + 6 + ... + \frac{n(n+1)}{2}$$

$$= 4n + \text{Sum of the first } n \text{ triangular numbers.}$$

$$= 4n + \frac{1}{2}(\Sigma n^2 + n)$$

$$= 4n + \frac{1}{2}(\Sigma n^2 = \Sigma n)$$

$$= 4n + \frac{1}{2}\left[\frac{n(n+1)(2n+10)}{6}\right] + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= 4n + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{1}{12}[48n + 2n(n+1)(n+2)]$$

$$= \frac{n}{6}[n^2 + 3n + 26].$$

Question 15. If a, b, c are any three integers, then show that $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ is divisible by 7.

Solution : Le us find the value of a^3 (mod 7) for any $a \in Z$.

As a (mod 7) is 0, 1, 2, 3, 4, 5 or 6, a^3 (mod 7) will be only among 0, 1 or 6.

Now, if 7 divides one of a, b, c, the given expression is divisible by 7. If not, then a^3, b^3, c^3 (mod 7) will be only among 1 and 6. Hence, two of them must be the same, say a^3 and b^3mod 7).

$\therefore$ $(a^3 - b^3)$ (mod 7) = 0. The given expression is divisible by 7.

Question 16. Show that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5.

The given expression may also be written as

$$(1^{99} + 4^{99}) + (2^{99} + 3^{99}) + 5^{99}$$

Now $\quad 1^{99} + 4^{99} = (1 + 4)(1^{98} + 1^{97}.4 + ... + 4^{98}) = 5 \times p$ (say)

Similarly $\quad 2^{99} + 3^{99} = 5 \times q$ say

$\therefore$ Expression is equal to

$$5p \times 5q + 5^{99} = 5 \times r \text{ (ay)}$$

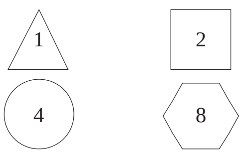
where p, q, r are integers

$\therefore$ 5 divides the given eumession as $5r$ is divisible by 5.

Question 17. Ram has four different coins with values as shown on the right. Suppose you had just one of each of these coins (4).

How many different amounts (value) can be made using one or more of the four different coins ? Explain.

Solution : Let us list the possible amounts :



1. Coin : This produces	1, 2, 4, 8		
2. Coin : They produce	3 = (1 + 2);	5 = (1 + 4);	9 = (1 + 8)
	6 = (2 + 4);	10 = (2 + 8);	12 = (4 + 8)
3. Coin : They produce	7 = (1 + 2 + 4),	11 = (1 + 2 + 8),	
	13 =(1 + 4 + 8);	14 = (2 + 4 + 8);	

4. Coin ; They produce $15 = (1 + 2 + 4 + 8)$

There are 15 different amounts that can be made.

These amounts are the first 15 counting numbers.

Question 18. Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162.

Solution : Let $19^{93} - 13^{99} = \theta$ (say)

Now $162 = (2) \times (81)$ since 19^{93} and 13^{99} are both odd one with add, their difference is even and hence divisible by 2 we have to show that θ divisible by 81.

Now $19^{93} \equiv (18+1)^{93} \equiv (93)18 + 1 \pmod{81}$

$\equiv 1675 \pmod{81} \equiv 55 \pmod{81}$

Also $13^{99} = (12+1)^{99} \equiv (99)12 + 1 \pmod{81}$

$= 1189 \pmod{81} \equiv 55 \pmod{81}$

$\therefore\ 19^{13} - 13^{99} \equiv 0 \pmod{81}$

Thus $162 = (2)(81)$ divides θ]

Question 19. Show that $19^{93} > 13^{99}$ without using tables/calculation.

Solution : $\left(\frac{19}{13}\right)^2 = \frac{361}{169} > 2;\ \left(\frac{19}{13}\right)^2 > 2^4 > 13;$

$19^8 > 13^9$

$\therefore$ $19^{88} > 13^{99}$

and $19^{93} > 13^{99}$.

Question 20. Let p be a prime number > 3.

What is the remainder when p^2 is divided by 12 ?

$p > 3$ and prime so p is odd.

$(p-1)$ and $(p+1)$ are both even

$2/p - 1$ and $2/p + 1$

i.e., $4/p^2 - 1$ …(1)

Also, $(p-1),\ p,\ (p+1)$ are consecutive integers.

$\therefore$ One of them must be divisible by 3. But $3/p$

$\therefore$ $3/p^2 - 1$ …(2)

$12/p^2 - 1$ from (1) and (2)

$\therefore$ when p^2 is divided by 12, the remainder is 1.