RELATIONS AND FUNCTIONS

CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 05 marks

QUICK REVISION (Important Concepts & Formulae)

Relation
Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.
R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

Total Number of Relations
Let A and B be two nonempty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of relations from A to B is 2^{nm}.

Domain and range of a relation
Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.
Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

Inverse relation
Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Types of Relations
Void relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A. It is the smallest relation on set A.

Universal relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A. It is the largest relation on set A.

Identity relation : Let A be a set. Then the relation $I\,A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R reflexive $\Leftrightarrow (a, a) \in R$, $a \in A$.

☞ A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Symmetric relation : A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$. *i.e.* $aRb \Rightarrow bRa$ for all $a, b \in A$.

☞ A relation R on a set A is not a symmetric relation if there are atleast two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \in R$.

Transitive relation : A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$. *i.e.* aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Antisymmetric relation : A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Equivalence relation : A relation R on a set A is said to be an equivalence relation on A iff
☞ It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$.

☞ It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
☞ It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Congruence modulo *m*
Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b(\text{mod } m)$. Thus, $a \equiv b \ (\text{mod } m) \Leftrightarrow \exists\ a - b$ is divisible by m.

Some Results on Relations
If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

☞ The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

☞ If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

Composition of relations
Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This relation is called the composition of R and S.

Functions
Let A and B be two empty sets. Then a function 'f ' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that
(i) All elements of set A are associated to elements in set B.
(ii) An element of set A is associated to a unique element in set B.
☞ A function 'f ' from a set A to a set B associates each element of set A to a unique element of set B.
☞ If an element $a \in A$ is associated to an element $b \in B$, then b is called 'the f image of a or 'image of a under f ' or 'the value of the function f at a'. Also, a is called the preimage of b under the function f. We write it as : $b = f(a)$.

Domain, CoDomain and Range of a function
Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the codomain of f. The set of all f images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$. Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f. Clearly, $f(A) \subseteq B$.

Equal functions
Two functions f and g are said to be equal iff
(i) The domain of f = domain of g
(ii) The codomain of f = the codomain of g, and
(iii) $f(x) = g(x)$ for every x belonging to their common domain.

☞ If two functions f and g are equal, then we write $f = g$.

Types of Functions
(i) Oneone function (injection)
A function $f : A \to B$ is said to be a oneone function or an injection if different elements of A have different images in B. Thus, $f : A \to B$ is oneone $\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A \Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

Algorithm to check the injectivity of a function
Step I : Take two arbitrary elements x, y (say) in the domain of f.
Step II : Put $f(x) = f(y)$
Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a oneone function (or an injection) otherwise not.

☞ Graphically, if any straight line parallel to x-axis intersects the curve $y = f(x)$ exactly at one point, then the function $f(x)$ is oneone or an injection. Otherwise it is not.

☞ If $f : R \to R$ is an injective map, then the graph of $y = f(x)$ is either a strictly increasing curve or a strictly decreasing curve. Consequently, $\frac{dy}{dx} > 0$ *or* $\frac{dy}{dx} < 0$ for all x.

☞ Number of oneone functions from A to B, $\begin{cases} {}^nP_m, & if \quad n \geq m \\ 0, & if \quad n < m \end{cases}$

where $m = n$(Domain) and $n = n$(Codomain)

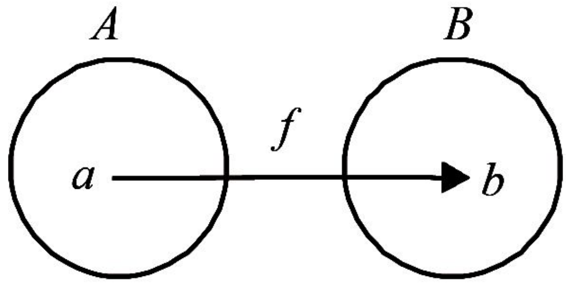
(ii) Ontofunction (surjection)
A function $f : A \to B$ is said to be an onto function or a surjection if every element of B is the $f$image of some element of A *i.e.*, if $f(A) = B$ or range of f is the codomain of f. Thus, $f : A \to B$ is a surjection iff for each $b \in B$, $\exists\ a \in A$ that $f(a) = b$.

Algorithm for Checking the Surjectivity of a Function
Let $f : A \to B$ be the given function.
Step I : Choose an arbitrary element y in B.
Step II : Put $f(x) = y$.
Step III : Solve the equation $f(x) = y$ for x and obtain x in terms of y. Let $x = g(y)$.
Step IV : If for all values of $y \in B$, for which x, given by $x = g(y)$ are in A, then f is onto.

If there are some $y \in B$ for which x, given by $x = g(y)$ is not in A. Then, f is not onto.

Number of onto functions : If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r} . {}^nC_r r^m$

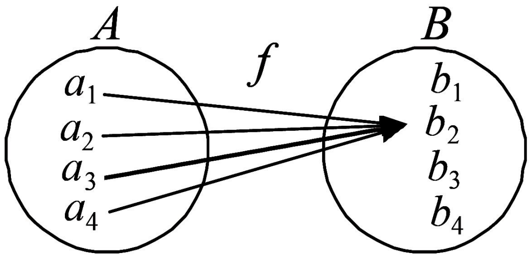
(iii) Bijection (oneone onto function)
A function $f : A \to B$ is a bijection if it is oneone as well as onto. In other words, a function $f : A \to B$ is a bijection if



(i) It is oneone *i.e.* $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.
(ii) It is onto *i.e.* for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Number of bijections : If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$

(iv) Manyone function
A function $f : A \to B$ is said to be a manyone function if two or more elements of set A have the same image in B.
$\therefore f : A \to B$ is a manyone function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.



(v) Into function

A function $f : A \to B$ is an into function if there exists an element in B having no preimage in A. In other words $f : A \to B$ is an into function if it is not an onto function.

(vi) Identity function

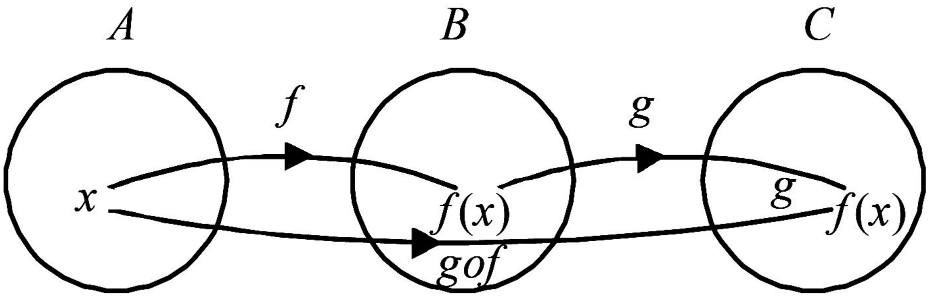
Let A be a nonempty set. A function $f : A \to A$ is said to be an identity function on set A if f associates every element of set A to the element itself. Thus $f : A \to A$ is an identity function iff $f(x) = x$, for all $x \in A$.

(vii) Constant function

A function $f : A \to B$ is said to be a constant function if every element of A has the same image under function of B *i.e.* $f(x) = c$ for all $x \in A$, where $c \in B$.

Composition of functions

Let A, B and C be three nonvoid sets and let $f : A \to B$, $g : B \to C$ be two functions. For each $x \in A$ there exists a unique element $g(f(x)) \in C$.



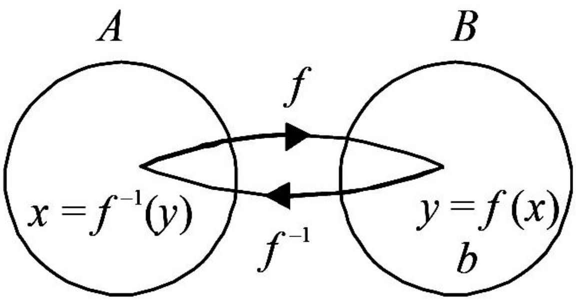
- The composition of functions is not commutative *i.e.* $fog \neq gof$.
- The composition of functions is associative *i.e.* if f, g, h are three functions such that $(fog)oh$ and $fo(goh)$ exist, then $(fog)oh = fo(goh)$.
- The composition of two bijections is a bijection *i.e.* if f and g are two bijections, then gof is also a bijection.
- Let $f : A \to B$. The $foI_A = I_B of = f$ *i.e.* the composition of any function with the identity function is the function itself.

Inverse of an element

Let A and B be two sets and let $f : A \to B$ be a mapping. If $a \in A$ is associated to $b \in B$ under the function f, then b is called the f image of a and we write it as $b = f(a)$.

Inverse of a function

If $f : A \to B$ is a bijection, we can define a new function from B to A which associates each element $y \in B$ to its preimage $f^{-1}(y) \in A$.



Algorithm to find the inverse of a bijection
Let $f : A \to B$ be a bijection. To find the inverse of f we proceed as follows :
Step I : Put $f(x) = y$, where $y \in B$ and $x \in A$.
Step II : Solve $f(x) = y$ to obtain x in terms of y.
Step III : In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the inverse of f.

Properties of Inverse of a Function
(i) The inverse of a bijection is unique.
(ii) The inverse of a bijection is also a bijection.
(iii) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.
If in the above property, we have $B = A$. Then we find that for every bijection $f : A \to A$ there exists a bijection $g : A \to A$ such that $fog = gof = I_A$.
(iv) Let $f : A \to B$ and $g : B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then f and g are bijections and $g = f^{-1}$.

Binary Operation
- ☞ Let S be a nonvoid set. A function f from $S \times S$ to S is called a binary operation on S *i.e.* $f : S \times S \to S$ is a binary operation on set S.
- ☞ Generally binary operations are represented by the symbols *, ⊕ ,. etc. instead of letters f, g etc.
- ☞ Addition on the set N of all natural numbers is a binary operation.
- ☞ Subtraction is a binary operation on each of the sets Z, Q, R and C. But, it is a binary operation on N.
- ☞ Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all nonzero rational (real or complex) numbers.

Types of Binary Operations
(i) Commutative binary operation
- ☞ A binary operation * on a set S is said to be commutative if $a * b = b * a$ for all $a, b \in S$
- ☞ Addition and multiplication are commutative binary operations on Z but subtraction is not a commutative binary operation, since $2 - 3 \neq 3 - 2$.
- ☞ Union and intersection are commutative binary operations on the power set $P(S)$ of all subsets of set S. But difference of sets is not a commutative binary operation on $P(S)$.

(ii) Associative binary operation
A binary operation * on a set S is said to be associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

(iii) Distributive binary operation
Let * and o be two binary operations on a set S. Then * is said to be
(i) Left distributive over o if $a*(b\ o\ c) = (a * b)\ o\ (a * c)$ for all $a, b, c \in S$
(ii) Right distributive over o if $(b\ o\ c) * a = (b * a)\ o\ (c * a)$ for all $a, b, c \in S$.

(iv) Identity element
- ☞ Let * be a binary operation on a set S. An element $e \in S$ is said to be an identity element for the binary operation * if $a * e = a = e * a$ for all $a \in S$.
- ☞ For addition on Z, 0 is the identity element, since $0 + a = a = a + 0$ for all $a \in R$.
- ☞ For multiplication on R, 1 is the identity element, since $1 \times a = a = a \times 1$ for all $a \in R$.
- ☞ For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

(v) Inverse of an element

☞ Let * be a binary operation on a set *S* and let *e* be the identity element in *S* for the binary operation *. An element $a' \in S$ is said to be an inverse of $a \in S$, if $a * a' = e = a' * a$.

☞ Addition on *N* has no identity element and accordingly *N* has no invertible element.

☞ Multiplication on *N* has 1 as the identity element and no element other than 1 is invertible.

☞ Let *S* be a finite set containing *n* elements. Then the total number of binary operations on *S* is n^{n^2} .

☞ Let *S* be a finite set containing *n* elements. Then the total number of commutative binary operation on *S* is $n\left[\frac{n(n+1)}{2}\right]$.

CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. **Determine whether each of the following relations are reflexive, symmetric and transitive : (iv) Relation R in the set Z of all integers defined as $R = \{(x, y): x - y$ is an integer $\}$**
   Ans:
   For reflexive put $y = x$, $x - x = 0$ which is an integer for all $x \in Z$. So, R is reflexive on Z.
   For symmetry let $(x,y) \in R$, then $(x - y)$ is an integer λ and also $y - x = -\lambda$ [$\therefore\ \lambda \in Z \Rightarrow -\lambda \in Z$]
   $\therefore\ y - x$ is an integer $\Rightarrow (y, x) \in R$. So, R is symmetric.
   For transitivity let $(x,y) \in R$ and $(y, z) \in R \Rightarrow x - y$ = integer and $y - z$ = integers, then $x - z$ is also an integer
   $\therefore (x, z) \in R$. So, R is transitive.

2. **Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.**
   Ans:
   We have $R = \{(a,b) : a \le b^2\}$, where $a, b \in R$
   For reflexivity, we observe that $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true.
   So, R is not reflexive as $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$
   For symmetry, we observe that $-1 \le 3^2$ but $3 > (-1)^2$
   $\therefore (-1, 3) \in R$ but $(3, -1) \notin R$.
   So, R is not symmetric.
   For transitivity, we observe that$2 \le (-3)^2$ and $-3 \le (1)^2$ but $2 > (1)^2$
   $\therefore (2, -3) \in R$ and $(-3,1) \in R$ but $(2, 1) \notin R$. So, R is not transitive.
   Hence, R is neither reflexive, nor symmetric and nor transitive.

3. **Show that the relation R in R defined as $R = \{a, b) : a \le b\}$, is reflexive and transitive but not symmetric.**
   Ans:
   We have $R = \{(a,b) : a \le b\}$. Let $a,b \in R$.
   Reflexive: for any $a \in R$ we have $a \le a$. So, R is reflexive.
   Symmetric: we observe that $(2, 3) \in R$ but $(3, 2) \notin R$. So, R is not symmetric.
   Transitivity: $(a,b) \in R$ and $(b, c) \in R \Rightarrow a \le b$ and $b \le c \Rightarrow a \le c \Rightarrow (a,c) \in R$
   So, R is transitive. Hence, R is reflexive and transitive but not symmetric.

4. **Check whether the relation R in R defined by $R = \{(a, b): a \le b^3\}$ is reflexive, symmetric or transitive.**
   Ans:
   Given that $R = \{(a, b): a \le b^3\}$
   It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \in R$ as $\frac{1}{2} < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
   So, R is not reflexive.
   Now, (1, 2) (as $1 < 2^3 = 8$)
   But $(2, 1) \notin$ R (as $2^3 > 1$)
   So, R is not symmetric.

We have $\left(3,\frac{3}{2}\right),\left(\frac{3}{2},\frac{6}{5}\right)\in R$ as $3>\left(\frac{3}{2}\right)^3$ *and* $\frac{3}{2}<\left(\frac{6}{5}\right)^3$

But $\left(3,\frac{6}{5}\right)\notin R\ \ as\ \ 3>\left(\frac{6}{5}\right)^3$

Therefore, *R* is not transitive.

Hence, *R* is neither reflexive nor symmetric nor transitive.

5. Show that the relation R in the set *A* = {1, 2, 3, 4, 5} given by R = {(a, b) : |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of 2, 4}.

Ans:

Given that A = {1, 2, 3, 4, 5} and R = {(a, b) : |a – b| is even}

It is clear that for any element $a \in A$, we have (which is even).

∴R is reflexive.

Let (*a*, *b*) ∈ R.

⇒ |a – b| is even

⇒(a – b) is even

⇒– (a – b) is even

⇒(b – a) is even

⇒ |b – a| is even

⇒(*b*, *a*) ∈ R

∴R is symmetric.

Now, let (*a*, *b*) ∈ R and (*b*, *c*) ∈ R.

⇒ |a – b| is even and |b – c| is even

⇒ (a – b) is even and (b – c) is even

⇒ (a – c) = (a – b) + (b – c) is even (Since, sum of two even integers is even)

⇒ |a – c| is even

⇒ (*a*, *c*) ∈ R

∴R is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set {1, 2, 3} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.

Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

6. Show that each of the relation R in the set $A=\{x\in Z:0\le x\le 12\}$, given by R = {(a, b): |a – b| is a multiple of 4} is an equivalence relation. Find the set of all elements related to 1.

Ans:

$A=\{x\in Z:0\le x\le 12\}=\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and

R = {(a, b): |a – b| is a multiple of 4}

For any element $a \in$A, we have (*a*, *a*) ∈ R ⇒ |a – a| = 0 is a multiple of 4.

∴R is reflexive.

Now, let (*a*, *b*) ∈ R ⇒|a – b| is a multiple of 4.

⇒|–(a – b)| is a multiple of 4

⇒|b – a| is a multiple of 4.

⇒ (*b*, *a*) ∈ R

∴ R is symmetric.
Now, let $(a, b), (b, c) \in$ R.
⇒|a – b| is a multiple of 4 and |b – c| is a multiple of 4.
⇒(a – b) is a multiple of 4 and (b – c) is a multiple of 4.
⇒(a – b + b – c) is a multiple of 4
⇒(a – c) is a multiple of 4
⇒|a – c| is a multiple of 4
⇒ $(a, c) \in$ R
∴ R is transitive.
Hence, R is an equivalence relation.
The set of elements related to 1 is {1, 5, 9} since
|1 – 1| = 0 is a multiple of 4
|5 – 1| = 4 is a multiple of 4
|9 – 1| = 8 is a multiple of 4

7. In each of the following cases, state whether the functions is one-one, onto or bijective. Justify answer.
(i) $f : R \to R$ defined by $f(x) = 3 - 4x$
(ii) $f : R \to R$ defined by $f(x) = 1 + x^2$
Ans:
(i) Here, $f : R \to R$ is defined by $f(x) = 3 - 4x$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\Rightarrow 3 - 4x_1 = 3 - 4x_2$
$\Rightarrow -4x_1 = -4x_2$
$\Rightarrow x_1 = x_2$
Therefore, f is one-one.
For any real number y in R, there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$
Therefore, f is onto. Hence, f is bijective.

(ii) Here $f : R \to R$ is defined as $f(x) = 1 + x^2$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\Rightarrow 1 + x_1^2 = 1 + x_2^2$
$\Rightarrow x_1^2 = x_2^2$
$\Rightarrow x_1 = \pm x_2$
For instance, $f(1) = f(-1) = 2$
Therefore, $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$
Therefore, f is not one-one.
Consider an element − 2 in co-domain R.
It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.
Thus, there does not exist any x in domain R such that $f(x) = -2$.
Therefore, f is not onto. Hence, f is neither one-one nor onto.

8. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$ is f one-one and onto ? Justify your answer.
Ans:
Here, $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$ is defined as $f(x) = \left(\frac{x-2}{x-3}\right)$
Let $x, y \in A$ such that $f(x) = f(y)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3} \Rightarrow (x-2)(y-3) = (y-2)(x-3)$
$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$
$\Rightarrow -3x - 2y = -3y - 2x$
$\Rightarrow 3x - 2x = 3y - 2y \Rightarrow x = y$

Therefore, f is one- one. Let $y \in B = R - \{1\}$. Then, $y \neq 1$
The function f is onto if there exists $x \in A$ such that $f(x) = y$.
Now, $f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y \Rightarrow x - 2 = xy - 3y$
$\Rightarrow x(1-y) = -3y + 2$
$\Rightarrow x = \frac{2-3y}{1-y} \in A \ \ [y \neq 1]$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1-y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

Therefore, f is onto. Hence, function f is one-one and onto.

9. If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, show that (fof)(x) = x, for all $x \neq \frac{2}{3}$. What is the inverse of f?

Ans:

Given that $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$

Then $(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Therefore (fof)(x) = x, for all $x \neq \frac{2}{3}$

Hence, the given function f is invertible and the inverse of f is itself.

10. Show that $f : [-1,1] \to R$, given by $f(x) = \frac{x}{x+2}, x \neq -2$, is one-one. Find the inverse of the function $f : [-1,1] \to$ Range f.

Ans:

Given that $f : [-1,1] \to R$, given by $f(x) = \frac{x}{x+2}, x \neq -2$,

Let f(x) = f(y)

$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2} \Rightarrow xy + 2x = xy + 2y$
$\Rightarrow 2x = 2y \Rightarrow x = y$

Therefore, f is a one-one function.

Let $y=\frac{x}{x+2} \Rightarrow x=xy+2y \Rightarrow x=\frac{2y}{1-y}$

So, for every *y* except 1 in the range there exists *x* in the domain such that *f*(*x*) = *y*. Hence, function *f* is onto.

Therefore, $f:[-1,1] \rightarrow$ Range *f* is one-one and onto and therefore, the inverse of the function $f:[-1,1] \rightarrow$ Range *f* exists.

Let *y* be an arbitrary element of range *f*.

Since, $f:[-1,1] \rightarrow$ Range *f* is onto, we have *y* = *f*(*x*) for some $x \in [-1,1]$

$\Rightarrow y=\frac{x}{x+2} \Rightarrow x=xy+2y \Rightarrow x=\frac{2y}{1-y}, y \neq 1$

Now, let us define *g*: Range $f \rightarrow [-1,1]$ as $g(y)=\frac{2y}{1-y}, y \neq 1$

$$(gof)(x)=g(f(x))=g\left(\frac{x}{x+2}\right)=\frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)}=\frac{2x}{x+2-x}=\frac{2x}{2}=x$$

and $(fog)(y)=f(g(y))=f\left(\frac{2y}{1-y}\right)=\frac{\left(\frac{2y}{1-y}\right)}{\left(\frac{2y}{1-y}\right)+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=y$

Therefore, *gof* = *fog* = I_R, Therefore, $f^{-1}=g$

Therefore, $f^{-1}(y)=\frac{2y}{1-y}, y \neq 1$

11. Consider *f* : *R*→ *R* given by *f* (*x*) = 4*x* + 3. Show that *f* is invertible. Find the inverse of *f* .

Ans:

Here, $f: R \rightarrow R$ is given by *f*(*x*) = 4*x* + 3

Let $x, y \in R$, such that *f*(*x*) = *f*(*y*)

$\Rightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y \Rightarrow x = y$

Therefore, *f* is a one-one function.

Let *y* = 4*x* + 3

$\Rightarrow$ There exist, $x=\left(\frac{y-3}{4}\right) \in R, \forall y \in R$

Therefore, for any $y \in R$, there exist $x=\left(\frac{y-3}{4}\right) \in R$ such that

$f(x)=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y$

Therefore, *f* is onto function.

Thus, *f* is one-one and onto and therefore, f^{-1} exists.

Let us define $g: R \rightarrow R$ by $g(x)=\frac{x-3}{4}$

Now, $(gof)(x)=g(f(x))=g(4x+3)=\frac{(4x+3)-3}{4}=x$

and $(fog)(y)=f(g(y))=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y$

Therefore, *gof* = *fog* = I_R

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \frac{y-3}{4}$

12. Consider f : $R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given by $f^{-1}y = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Ans: Here, function $f : R_+ \to [4,\infty]$ is given as $f(x) = x^2 + 4$

Let $x, y \in R_+$, such that $f(x) = f(y)$

$\Rightarrow x^2 + 4 = y^2 + 4 \Rightarrow x^2 = y^2$

$\Rightarrow x = y$ [as $x = y \in R_+$]

Therefore, f is a one-one function.

For $y \in [4,\infty)$, let $y = x^2 + 4$

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in R_+$, There exists $x = \sqrt{y-4} \in R_+$ such that

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$

Therefore, f is onto. Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g : [4,\infty) \to R_+$ by $g(y) = \sqrt{y-4}$

Now, $gof(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$

and $fog(y) = f(g(y)) = f(\sqrt{y-4}) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$

Therefore, $gof = I_{R_+}$ and $fog = I_{[4,\infty)}$

Hence, f is invertible and the inverse of f if given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

13. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$.

Ans:

Here, function $f : R_+ \to [-5,\infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be any arbitrary element of $[-5,\infty)$.

Let $y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x + 1)^2 - 1 - 5 = (3x + 1)^2 - 6$

$\Rightarrow (3x + 1)^2 = y + 6$

$\Rightarrow (3x + 1)^2 = \sqrt{y+6}$ [as $y \geq -5 \Rightarrow y + 6 \geq 0$]

$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

Therefore, f is onto, thereby range $f = [-5,\infty)$.

Let us define $g : [-5,\infty) \to R_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$

Now, $(gof)(x) = g(f(x)) = g(9x^2 + 6x - 5) = g((3x + 1)^2 - 6)$

$= \frac{\sqrt{(3x+1)^2-6+6}-1}{3} = \frac{3x+1-1}{3} = x$

and $(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right) = \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$

$= \left(\sqrt{y+6}\right)^2 - 6 = y + 6 - 6 = y$

Therefore, $gof = I_{R_+}$ and $fog = I_{[-5,\infty)}$

Hence, f is invertible and the inverse of f if given by $f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}$

14. Let * be the binary operation on N given by $a * b$ = LCM of a and b.
(i) Find 5*7 , 20*16 (ii) Is * commutative?
(iii) Is * associative? (iv) Find the identity of * in N
(v) Which elements of N are invertible for the operation *?
Ans:
The binary operation on N is defined as $a*b$ = LCM of a and b.
(i) We have 5 *7 = LCM of 5 and 7 = 35 and 20*16 = LCM of 20 and 16 = 80

(ii) It is known that LCM of a and b = LCM of b and a for $a, b \in N$.
Therefore, $a*b = b*a$. Thus, the operation * is commutative.

(iii) For $a, b, c \in N$, we have
$(a*b) *c$ = (LCM of a and b) $*c$ = LCM of a, b, and c
$a* (b*c) = a$ * (LCM of b and c) = LCM of a, b, and c
Therefore, $(a*b) *c = a* (b*c)$. Thus, the operation is associative.

(iv) It is known that LCM of a and 1 = a = LCM of 1 and a, $a \in N$.
$a*1 = a = 1*a$, $a \in N$
Thus, 1 is the identity of * in N.

(v) An element a in N is invertible with respect to the operation *, if there exists an element b in N such that $a*b = e = b*a$.
Here, $e = 1$. This means that
LCM of a and b = 1 = LCM of b and a
This case is possible only when a and b are equal to 1.
Thus, 1 is the only invertible element of N with respect to the operation *.

15. Let * be the binary operation on N defined by $a*b$ =HCF of a and b. Is * commutative? Is * associative? Does there exist identity for this binary operation on N?
Ans:
The binary operation * on N is defined as $a*b$ = HCF of a and b.
It is known that HCF of a and b = HCF of b and a for $a,b \in N$.
Therefore, $a*b = b*a$. Thus, the operation is commutative.
For $a,b,c \in N$, we have $(a*b)*c$ = (HCF of a and b)$*c$ = HCF of a,b and c
$a*(b*c) = a*$(HCF of b and c) = HCF of a,b, and c
Therefore, $(a*b)*c = a*(b*c)$
Thus, the operation * is associative.
Now, an element $e \in N$ will be the identity for the operation if $a*e = a = e*a, \forall\ a \in N$.
But this relation is not true for any $a \in N$.
Thus, the operation * does not have identity in N.

16. Let * be a binary operation on the set Q of rational number as follows :
(i) $a*b = a - b$ (ii) $a*b = a^2 + b^2$ (iii) $a*b = a + ab$
(iv) $a*b = (a - b)^2$ (v) $a*b = \frac{ab}{4}$ (vi) $a*b = ab^2$
Find which of the binary operation are commutative and which are associative?
Ans:
(i) On Q, the operation * is defined as $a*b = a - b$.
It can be observed that for $2,3,4 \in Q$, we have $2*3 = 2 - 3 = -1$ and $3*2 = 3 - 2 = 1$
$\Rightarrow 2*3 \neq 3*2$
Thus, the operation is not commutative.

It can also be observed that
(2*3)*4 = (−1)*4 = −1 − 4 = − 5
and 2*(3*4) = 2*(−1) = 2 − (−1) = 3
2*(3*4) $\neq$ 2* (3*4)
Thus, the operation * is not associative.

(ii) On Q, the operation * is defined as $a*b = a^2 + b^2$.
For $a,b \in Q$, we have
$a*b = a^2 + b^2 = b^2 + a^2 = b*a$
Therefore, $a*b = b*a$
Thus, the operation * is commutative.
It can be observed that
$(1* 2)* 3 = (1^2 + 2^2) * 3$ $(1 + 4) * 4 = 5 * 4 = 5^2 + 4^2 = 41$
and $1*(2* 3) = 1* (2^2 + 3^2) = 1 * (4 + 9) = 1*13 = 1^2 + 13^2 = 170$
$\Rightarrow$ (1*2)* 3 $\neq$ 1*(2*3) where 1,2, 3 $\in Q$
Thus, the operation * is not associative.

(iii) On Q, the operation is defined as $a*b = a + ab$
It can be observed that
1*2 = 1 + 1 × 2 = 1 + 2 = 3, 2*1 = 2 + 2 × 1 = 2 + 2 = 4
$\Rightarrow$ 1*2 $\neq$ 2*1 where 1, 2 $\in Q$
Thus, the operation * is not commutative.
It can also be observed that
(1*2)*3 = (1 + 1 × 2)*3 = 3*3 = 3 + 3 × 3 = 3 + 9 = 12
and 1*(2* 3) = 1*(2 + 2 × 3) = 1*8 = 1 + 1 × 8 = 9
$\Rightarrow$ (1*2)* 3 $\neq$ 1*(2*3) where 1,2, 3 $\in Q$
Thus, the operation * is not associative.

(iv) On Q, the operation * is defined by $a*b = (a - b)^2$.
For $a,b \in Q$, we have $a*b = (a - b)^2$ and $b*a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$
Therefore, $a*b = b*a$
Thus, the operation * is commutative. It can be observed that
$(1* 2) * 3 = (1 - 2)^2 * 3 = (1) * 3 = (1 - 3)^2 = 4$
and $1 * (2 *3) = 1 * (2 - 3)^2 = 1 * (1) = (1 - 1)^2 = 0$
$\Rightarrow$ (1*2)* 3 $\neq$ 1*(2* 3) where 1,2, 3 $\in Q$
Thus, the operation * is not associative.

(v) On Q, the operation * is defined as $a*b = \frac{ab}{4}$

For $a,b \in Q$, we have $a * b = \frac{ab}{4} = \frac{ba}{4} = b*a$

Therefore, $a*b = b*a$
Thus, the operation * is commutative.

For $a,b,c \in Q$, we have $a*(b*c) = \frac{ab}{4}*c = \frac{\frac{ab}{4}.c}{4} = \frac{abc}{16}$ and $a*(b*c) = a*\frac{bc}{4} = \frac{a.\frac{bc}{4}}{4} = \frac{abc}{16}$
Therefore, $(a*b)*c = a*(b*c)$. Thus, the operation * is associative.

(vi) On Q, the operation is defined as $a*b = ab^2$
It can be observed that for 23 $\in Q$
$2 * 3 = 2 \text{ x } 3^2 = 18$ and $3*2 = 3 \text{ x } 2^2 = 12$
Hence, 2*3 $\neq$ 3*2
Thus, the operation is not commutative.

It can also be observed that for 1,2,3 $\in Q$
(1*2)*3 = (1. 2^2)*3 = 4 *3 = 4.3^2 = 36
and 1*(2* 3) = 1*(2. 3^2) = 1 *18 =1.18^2 = 324
$\Rightarrow$ (1*2)*3 $\neq$ 1*(2*3)
Thus, the operation * is not associative.
Hence, the operations defined in parts (ii), (iv), (v) are commutative and the operation defined in part (v) is associative.

17. Show that none of the operation given in the above question has identity.
Ans:
An element $e \in Q$ will be the identity element for the operation if
$a*e = a = e*a,\ a \in Q$
(i) $a*b = a - b$
If $a*e = a, a \neq 0 \Rightarrow a - e = a, a \neq 0 \Rightarrow e = 0$
Also, $e*a = a \Rightarrow e - a = a \Rightarrow e = 2a$
$\Rightarrow e = 0 = 2a, a \neq 0$
But the identiry is unique. Hence this operation has no identity.

(ii) $a*b = a^2 + b^2$
If $a*e = a$, then $a^2 + e^2$ = a
For $a = -2$, $(-2)^2 + e^2 = 4 + e^2 \neq -2$
Hence, there is no identity element.

(iii) $a*b = a + ab$
If $a*e = a \Rightarrow a + ae = a \Rightarrow ae = 0 \Rightarrow e = 0, a \neq 0$
Also if $e*a = a \Rightarrow e + ea = a \Rightarrow e = \frac{a}{1-a}, a \neq 1$

$\therefore e = 0 = \frac{a}{1-a}, a \neq 0$
But the identity is unique. Hence this operation has no identify.

(iv) $a*b = (a - b)^2$
If $a*e = a$, then $(a - e)^2 = a$. A square is always positive, so for
$a = -2, (-2 - e)^2 \neq -2$
Hence, there is no identity element.

(v) $a*b = ab/4$
If $a*e = a$, then $ae/4 = a$. Hence, $e = 4$ is the identity element.
$\therefore a*4 = 4*a = 4a/4 = a$

(vi) $a*b = ab^2$
If $a*e = a$ then $ae^2 = a \Rightarrow e^2 = 1 \Rightarrow e = \pm 1$
But identity is unique. Hence this operation has no identity.
Therefore only part (v) has an identity element.

18. Show that the function $f : R \rightarrow \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one-one and onto function.
Ans:
It is given that $f : R \rightarrow \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$

Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$

It can be observed that if x is positive and y is negative, then we have

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative, then $x > y \Rightarrow x - y > 0$

But, $2xy$ is negative. Then, $2xy \neq x - y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out. Therefore, x and y have to be either positive or negative.

When x and y are both positive, we have $f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$

When x and y are both negative, we have $f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$

Therefore, f is one-one. Now, let $y \in R$ such that $-1 < y < 1$.

If y is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

Therefore, f is onto. Hence, f is one-one and onto.

19. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective.

Ans:

Here, $f : R \to R$ is given as $f(x) = x^3$.

Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow x^3 = y^3$ …(i)

Now, we need to show that $x = y$

Suppose, $x \neq y$, their cubes will also not be equal.

$x^3 \neq y^3$

However, this will be a contradiction to Eq. i).

Therefore, $x = y$. Hence, f is injective.

20. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as $a * b = \begin{cases} a+b, & if a+b < 6 \\ a+b-6, & if a-b \geq 6 \end{cases}$. Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $(6 - a)$ being the inverse of a.

Ans:

Let $X = \{0,1,2,3,4,5\}$

The operation * on X is defined as $a * b = \begin{cases} a+b, & if a+b < 6 \\ a+b-6, & if a-b \geq 6 \end{cases}$

An element $e \in X$ is the identity element for the operation *, if
$a*e = a = e*a \ \forall a \in X$
For $a \in X$, we observed that
$a*0 = a + 0 = a \quad [\because a \in X \Rightarrow a + 0 < 6]$
$0*a = 0 + a = a \quad [\because a \in X \Rightarrow 0 + a < 6]$
$\therefore a*0 = a = 0*a \ \forall a \in X$

Thus, 0 is the identity element for the given operation *.
An element $a \in X$ is invertible, if there exists $b \in X$ such that $a*b = 0 = b*a$

$$i.e. \begin{cases} a+b=0=b+a, & if a+b<6 \\ a+b-6=0=b+a-6, & if a-b \geq 6 \end{cases}$$

i.e., $a = -b$ or $b = 6 - a$
But $X = \{0,1,2,3,4,5\}$ and $a, b \in X$
Then, $a \neq -b$
Therefore, $b = 6 - a$ is the inverse of a, $a \in X$.
Hence, the inverse of an element $a \in X$, $a \neq 0$ is $(6 - a)$ *i.e.*, $a^{-1} = 6 - a$.

21. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides $a – b$} is an equivalence relation.
Ans:
R is reflexive, as 2 divides $(a – a)$ for all $a \in \mathbf{Z}$.
Further, if $(a, b) \in$ R, then 2 divides $a – b$.
Therefore, 2 divides $b – a$.
Hence, $(b, a) \in$ R, which shows that R is symmetric.
Similarly, if $(a, b) \in$ R and $(b, c) \in$ R, then $a – b$ and $b – c$ are divisible by 2.
Now, $a – c = (a – b) + (b – c)$ is even.
So, $(a – c)$ is divisible by 2. This shows that R is transitive.
Thus, R is an equivalence relation in **Z**.

22. Show that if f : A → B and g : B → C are one-one, then gof : A → C is also one-one.
Ans:
Suppose $gof(x_1) = gof(x_2)$
$\Rightarrow g(f(x_1)) = g(f(x_2))$
$\Rightarrow f(x_1) = f(x_2)$, as g is one-one
$\Rightarrow x_1 = x_2$, as f is one-one
Hence, gof is one-one.

23. Determine which of the following binary operations on the set N are associative and which are commutative. (a) $a * b = 1 \ \forall a,b \in N$ (b) $a * b = \frac{a+b}{2} \forall a,b \in N$

Ans:
(a) Clearly, by definition $a * b = b * a = 1 \ \forall a,b \in N$.
Also $(a * b) * c = (1 * c) = 1$ and $a * (b * c) = a * (1) = 1, \forall a,b,c \in N$.
Hence R is both associative and commutative.
(b) $a*b = \frac{a+b}{2} = \frac{b+a}{2} = b*a$, shows that * is commutative. Further,

$$(a*b)*c = \left(\frac{a+b}{2}\right)*c = \frac{\left(\frac{a+b}{2}\right)+c}{2} = \frac{a+b+2c}{4}$$

But $a*(b*c) = a*\left(\frac{b+c}{2}\right) = \frac{a+\left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4}$ in general.

Hence, * is not associative.

24. Show that if $f : R-\left\{\frac{7}{5}\right\} \to R-\left\{\frac{3}{5}\right\}$ is defined by $f(x) = \frac{3x+4}{5x-7}$ and $g : R-\left\{\frac{3}{5}\right\} \to R-\left\{\frac{7}{5}\right\}$ is defined by $g(x) = \frac{7x+4}{5x-3}$, then *fog* = I_A and *gof* = I_B, where, $A = R-\left\{\frac{3}{5}\right\}, B = R-\left\{\frac{7}{5}\right\}$; $I_A(x) = x$, $\forall\ x \in$ A, $I_B(x) = x$, $\forall\ x \in$ B are called identity functions on sets A and B, respectively.

Ans:

We have $gof(x) = g\left(\frac{3x+4}{5x-7}\right) = \frac{7\left(\frac{3x+4}{5x-7}\right)+4}{5\left(\frac{3x+4}{5x-7}\right)-3} = \frac{21x+28+20x-28}{15x+20-15x+21} = \frac{41x}{41} = x$

Similarly, $fog(x) = f\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} = \frac{21x+12+20x-12}{35x+20-35+21} = \frac{41x}{41} = x$

Thus, *gof* (x) = x, $\forall\ x \in$ B and *fog* (x) = x, $\forall\ x \in$ A, which implies that *gof* = I_B and *fog* = I_A.

25. Let *f* : N →R be a function defined as *f* (x) = $4x^2 + 12x + 15$. Show that *f* : N→ S, where, S is the range of *f*, is invertible. Find the inverse of *f*.

Ans:

Let y be an arbitrary element of range *f*. Then $y = 4x^2 + 12x + 15$, for some x in **N**, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{\sqrt{y-6}-3}{2}$, as $y \geq 6$.

Let us define g : S → **N** by $g(y) = \frac{\sqrt{y-6}-3}{2}$

Now *gof* (x) = $g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$$= \frac{\sqrt{(2x+3)^2+6-6}-3}{2} = \frac{(2x+3-3)}{2} = x$$

and $fog(y) = f\left(\frac{\sqrt{y-6}-3}{2}\right) = \left(\left(2\left(\frac{\sqrt{y-6}-3}{2}\right)+3\right)^2+6\right)$

$$= \left(\sqrt{y-6}-3+3\right)^2+6 = \left(\sqrt{y-6}\right)^2+6 = y-6+6 = y$$

Hence, *gof* = $I_{\mathbf{N}}$ and *fog* = I_S. This implies that *f* is invertible with $f^{-1} = g$.

CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. If $f(x) = x + 7$ and $g(x) = x – 7$, $x \in R$, find $(fog)(7)$

Ans:

Given $f(x) = x + 7$ and $g(x) = x – 7$, $x \in R$

$fog(x) = f(g(x)) = g(x) + 7 = (x – 7) + 7 = x$

$\Rightarrow (fog)(7) = 7$.

2. **If $f(x)$ is an invertible function, find the inverse of $f(x) = \frac{3x-2}{5}$**

Ans:

Given $f(x) = \frac{3x-2}{5}$

$Let \ \ y = \frac{3x-2}{5}$

$\Rightarrow 3x - 2 = 5y \Rightarrow x = \frac{5y+2}{3}$

$\Rightarrow f^{-1}(x) = \frac{5x+2}{3}$

3. Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

Ans:

(*i*) Reflexive

R is reflexive if $T_1 \ R \ T_1$

Since $T_1 \cong T_1$

$\therefore$ R is reflexive.

(*ii*) Symmetric

R is symmetric if $T_1 \ R \ T_2 \Rightarrow T_2 \ R \ T_1$

Since $T_1 \cong T_2 \Rightarrow T_2 \cong T_1$

$\therefore$ R is symmetric.

(*iii*) Transitive

R is transitive if $T_1 \ R \ T_2$ and $T_2 \ R \ T_3 \Rightarrow T_1 \ R \ T_3$

Since $T_1 \cong T_2$ and $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$

$\therefore$ R is transitive

From (*i*), (*ii*) and (*iii*), we get R is an equivalence relation.

4. If the binary operation * on the set of integers Z, is defined by $a * b = a + 3b^2$, then find the value of 2 * 4.

Ans:

Given $a * b = a + 3b^2 \ \ \forall a, b \in z$

$\therefore$ $2*4 = 2 + 3 \times 4^2 = 2 + 48 = 50$

5. Let * be a binary operation on N given by $a * b$ = HCF (a, b) $a, b \in N$. Write the value of 22 * 4.

Ans:

Given $a * b$ = HCF (a, b), $a, b \in N$

$\Rightarrow$ 22 * 4 = HCF (22, 4) = 2

6. **Let $f : N \rightarrow N$ be defined by** $f(n)=\begin{cases}\frac{n+1}{2}, if \quad n \quad is \quad odd \\ \frac{n}{2}, if \quad n \quad is \quad even\end{cases}$ $for \quad all \quad n \in N$ **. Find whether the function *f* is bijective.**

Ans:

Given that $f : N \rightarrow N$ be defined by $f(n)=\begin{cases}\frac{n+1}{2}, if \quad n \quad is \quad odd \\ \frac{n}{2}, if \quad n \quad is \quad even\end{cases}$ $for \quad all \quad n \in N$.

Let $x, y \in N$ and let they are odd then

$f(x)=f(y) \Rightarrow \frac{x+1}{2}=\frac{y+1}{2} \Rightarrow x=y$

If $x, y \in N$ are both even then also

$f(x)=f(y) \Rightarrow \frac{x}{2}=\frac{y}{2} \Rightarrow x=y$

If $x, y \in N$ are such that x is even and y is odd then

$f(x)=\frac{x+1}{2} \quad and \quad f(y)=\frac{y}{2}$

Thus, $x \neq y$ for $f(x) = f(y)$

Let $x = 6$ and $y = 5$

We get $f(6)=\frac{6}{2}=3, f(5)=\frac{5+1}{2}=3$

$\therefore$ $f(x) = f(y)$ but $x \neq y$ ...(*i*)

So, $f(x)$ is not one-one.

Hence, $f(x)$ is not bijective.

7. **If the binary operation *, defined on Q, is defined as $a * b = 2a + b - ab$, for all $a, b \in Q$, find the value of 3 * 4 .**

Ans:

Given binary operation is $a*b = 2a + b - ab$

$\therefore$ $3*4 = 2 \times 3 + 4 - 3 \times 4$

$\Rightarrow$ $3*4 = -2$

8. **What is the range of the function** $f(x)=\frac{|x-1|}{(x-1)}$**?**

Ans:

We have given $f(x)=\frac{|x-1|}{(x-1)}$

$|x-1|=\begin{cases}(x-1), \ if \quad x-1>0 \quad or \quad x>1 \\ -(x-1), if \quad x-1<0 \quad or \quad x<1\end{cases}$

(*i*) For $x > 1$, $f(x)=\frac{(x-1)}{(x-1)}=1$

(*ii*) For $x < 1$, $f(x)=\frac{-(x-1)}{(x-1)}=-1$

$\therefore$ Range of $f(x)=\frac{|x-1|}{(x-1)}$ is $\{-1, 1\}$.

9. Let Z be the set of all integers and R be the relation on Z defined as $R = \{(a, b) ; a, b \in Z$, and $(a - b)$ is divisible by 5.$\}$ Prove that R is an equivalence relation.

Ans:

We have provided $R = \{(a, b) : a, b \in Z$, and$(a - b)$ is divisible by 5$\}$

(i) As $(a - a) = 0$ is divisible by 5.

$\therefore (a, a) \in R \ \forall \ a \in R$

Hence, R is reflexive.

(ii) Let $(a, b) \in R$

$\Rightarrow (a - b)$ is divisible by 5.

$\Rightarrow -(b - a)$ is divisible by 5.

$\Rightarrow (b - a)$ is divisible by 5.

$\therefore (b, a) \in R$

Hence, R is symmetric.

(iii) Let $(a, b) \in R$ and $(b, c) \in Z$

Then, $(a - b)$ is divisible by 5 and $(b - c)$ is divisible by 5.

$(a - b) + (b - c)$ is divisible by 5.

$(a - c)$ is divisible by 5.

$\therefore (a, c) \in R$

$\Rightarrow R$ is transitive.

Hence, R is an equivalence relation.

10. Let * be a binary operation on Q defined by $a * b = \dfrac{3ab}{5}$. Show that * is commutative as well as associative. Also find its identity element, if it exists.

Ans:

For commutativity, condition that should be fulfilled is $a * b = b * a$

Consider $a * b = \dfrac{3ab}{5} = \dfrac{3ba}{5} = b * a$

$\therefore a * b = b * a$

Hence, * is commutative.

For associativity, condition is $(a * b) * c = a * (b * c)$

Consider $(a * b) * c = \left(\dfrac{3ab}{5}\right) * c = \dfrac{9ab}{25}$

and $a * (b * c) = a * \left(\dfrac{3bc}{5}\right) = \dfrac{9ab}{25}$

Hence, $(a * b) * c = a * (b * c)$

$\therefore$ * is associative.

Let $e \in Q$ be the identity element,

Then $a * e = e * a = a$

$\Rightarrow \dfrac{3ae}{5} = \dfrac{3ea}{5} = a \Rightarrow e = \dfrac{5}{3}$

11. If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.

Ans:

If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$ then $(fof)\ x = f(f(x)) = f[(3 - x^3)^{1/3}]$

$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3} = [3 - (3 - x^3)]^{1/3} = (x^3)^{1/3} = x$

12. Let $A = N ´ N$ and * be a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Also, find the identity element for * on A, if any.

Ans:

Given $A = N \times N$

* is a binary operation on A defined by

$(a, b) * (c, d) = (a + c, b + d)$

(i) Commutativity: Let $(a, b), (c, d) \in N \times N$

Then $(a, b) * (c, d) = (a + c, b + d) = (c + a, d + b)$

($\because a, b, c, d \in$ N, $a + c = c + a$ and $b + d = d + c$)

$= (c, d) * b$

Hence, $(a, b) * (c, d) = (c, d) * (a, b)$

$\therefore$ * is commutative.

(ii) Associativity: let $(a, b), (b, c), (c, d)$

Then $[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) = ((a + c) + e, (b + d) + f)$

$= \{a + (c + e), b + (d + f)]$ ($\because$ set N is associative)

$= (a, b) * (c + e, d + f) = (a, b) * \{(c, d) * (e, f)\}$

Hence, $[(a, b) * (c, d)] * (e, f) = (a, b) * \{(c, d) * (e, f)\}$

$\therefore$ * is associative.

(iii) Let (x, y) be identity element for $\forall$ on A,

Then $(a, b) * (x, y) = (a, b)$

$\Rightarrow (a + x, b + y) = (a, b)$

$\Rightarrow a + x = a, b + y = b$

$\Rightarrow x = 0, y = 0$

But $(0, 0) \notin A$

$\therefore$ For *, there is no identity element.

13. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, find $gof(x)$.

Ans:

Given $f : R \to R$ and $g : R \to R$ defined by $f(x) = \sin x$ and $g(x) = 5x^2$

$\therefore gof(x) = g[f(x)] = g(\sin x) = 5(\sin x)^2 = 5\sin^2 x$

14. Consider the binary operation* on the set {1, 2, 3, 4, 5} defined by a * b = min. {a, b}. Write the operation table of the operation *.

Ans:

Required operation table of the operation * is given as

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

15. If $f : R \to R$ is defined by $f(x) = 3x + 2$, define $f[f(x)]$.

Ans:

$f(f(x)) = f(3x + 2)$

$=3. (3x + 2) + 2 = 9x + 6 + 2$

$= 9x + 8$

16. Write *fog*, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

Ans:

$fog\ (x) = f\ (g(x))$

$= f\ (x^{1/3})$

$= 8(x^{1/3})^3$

$= 8x$

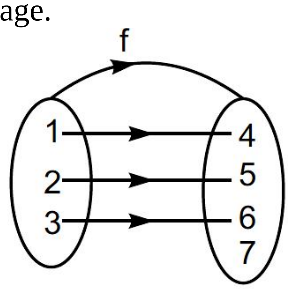
17. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let *f* = {(1, 4), (2,5), (3, 6)} be a function from A to B. State whether *f* is one-one or not.

Ans:

f is one-one because

$f(1) = 4$; $f(2) = 5$; $f(3) = 6$

No two elements of *A* have same *f* image.



18. Let $f : R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \to R$ such that gof = fog = I_R.

Ans:

$\because gof = fog = IR$

$\Rightarrow fog = IR$

$\Rightarrow fog(x) = I\ (x)$

$\Rightarrow f\ (g(x)) = x$ [$\because I(x) = x$ being identity function]

$\Rightarrow 10(g(x)) + 7 = x$ [$\because f(x) = 10x + 7$]

$\Rightarrow g(x) = \dfrac{x-7}{10}$

i.e., $g : R \to R$ is a function defined as $g(x) = \dfrac{x-7}{10}$

19. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$.

Show that *f* is one-one and onto and hence find f^{-1}.

Ans:

Let $x_1, x_2 \in A$.

Now, $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1-2}{x_1-3} = \dfrac{x_2-2}{x_2-3}$

$\Rightarrow (x_1-2)(x_2-3) = (x_1-3)(x_2-2)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2$

$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

Hence *f* is one-one function.

For Onto

Let $y = \dfrac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$

$\Rightarrow xy - x = 3y - 2 \Rightarrow x(y-1) = 3y - 2$

$\Rightarrow x = \dfrac{3y-2}{y-1}$ - - - - - (i)

From above it is obvious that $\forall$ y except 1, *i.e.*, $\forall y \in B = R - \{1\}$ $\exists x \in A$
Hence f is onto function.
Thus f is one-one onto function.
It f^{-1} is inverse function of f then $f^{-1}(y) = \dfrac{3y-2}{y-1}$ [from (*i*)]

20. The binary operation * : $R \times R \to R$ is defined as $a * b = 2a + b$. Find (2 * 3) * 4
Ans:
(2 * 3) * 4 = (2 × 2 +3) * 4
= 7 * 4
= 2 × 7 + 4 = 18

21. Show that f : N → N, given by $f(x) = \begin{cases} x+1, if \quad x \ is \ odd \\ x-1, if \quad x \ is \ even \end{cases}$ **is both one-one and onto.**

Ans:
For one-one
Case I : When x_1, x_2 are odd natural number.
$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1+1 = x_2 +1$ $\forall$ x_1 , $x_2 \in N$
$\Rightarrow x_1 = x_2$
i.e., f is one-one.

Case II : When x_1, x_2 are even natural number
$\therefore f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$
$\Rightarrow x_1 = x_2$
i.e., f is one-one.

Case III : When x_1 is odd and x_2 is even natural number
$f(x_1) = f(x_2) \Rightarrow x_1+1 = x_2 - 1$
$\Rightarrow x_2 - x_1 = 2$ which is never possible as the difference of odd and even number is always odd number.
Hence in this case $f(x_1) \neq f(x_2)$
i.e., f is one-one.

Case IV: When x_1 is even and x_2 is odd natural number
Similar as case III, We can prove f is one-one
For onto:
$\therefore f(x) = x + 1$ if x is odd
$= x - 1$ if x is even
$\Rightarrow$ For every even number 'y' of codomain $\exists$ odd number y - 1 in domain and for every odd number y of codomain $\exists$ even number y +1 in Domain.
i.e. f is onto function.
Hence f is one-one onto function.

22. Consider the binary operations * : $R \times R \to$ R and o : $R \times R \to R$ defined as $a * b = |a - b|$ and $aob = a$ for all $a, b \in R$. Show that '*' is commutative but not associative, 'o' is associative but not commutative.
Ans:
For operation '*'
'*' : R ´ $R \to R$ such that
$a*b = |a - b|$ $\forall a, b \in R$
Commutativity
$a*b = |a - b| = |b - a| = b * a$
i.e., '*' is commutative

Associativity

$\forall\ a, b, c \in R$

$(a * b) * c = |a - b| * c = ||a - b| - c|$

$a * (b * c) = a * |b - c| = |a - |b - c||$

But $||a - b| - c| \neq |a - |b - c||$

$\Rightarrow (a*b)* c \neq a*(b * c)$ " $a, b, c \in R$

$\Rightarrow$ * is not associative.

Hence, '*' is commutative but not associative.

For Operation 'o'

$o : R \times R \to R$ such that $aob = a$

Commutativity $\forall\ a, b \in R$

$aob = a$ and $boa = b$

$\because a \neq b \Rightarrow aob \neq boa$

$\Rightarrow$ 'o' is not commutative.

Associativity: " $a, b, c \in R$

$(aob)\ oc = aoc = a$

$ao(boc) = aob = a$

$\Rightarrow (aob)\ oc = ao\ (boc)$

$\Rightarrow$ 'o' is associative

Hence 'o' is not commutative but associative.

23. If the binary operation * on the set Z of integers is defined by $a * b = a + b - 5$, then write the identity element for the operation * in Z.

Ans:

Let e $\in Z$ be required identity

$\therefore\ a * e = a\ \forall\ a \in Z$

$\Rightarrow a + e - 5 = a$

$\Rightarrow e = a - a + 5$

$\Rightarrow e = 5$

24. If the binary operation * on set R of real numbers is defined as $a*b = \dfrac{3ab}{7}$, write the identity element in R for *.

Ans:

Let $e \in R$ be identity element.

$\therefore\ a * e = a\ \forall\ a \in R$

$\Rightarrow \dfrac{3ae}{7} = a \Rightarrow e = \dfrac{7a}{3a}$

$\Rightarrow e = \dfrac{7}{3}$

25. Prove that the relation R in the set $A = \{5, 6, 7, 8, 9\}$ given by $R = \{(a, b) : |a - b|$, is divisible by 2}, is an equivalence relation. Find all elements related to the element 6.

Ans:

Here R is a relation defined as $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$

Reflexivity

Here $(a, a) \in R$ as $|a - a| = 0 = 0$ divisible by 2 *i.e.*, R is reflexive.

Symmetry

Let $(a, b) \in R$

$(a, b) \in R \Rightarrow |a - b|$ is divisible by 2

$\Rightarrow a - b = \pm 2m \Rightarrow b - a = \mp 2m$

$\Rightarrow |b - a|$ is divisible by 2 $\Rightarrow (b, a) \in R$

Hence R is symmetric

Transitivity Let $(a, b), (b, c) \in R$

Now, $(a, b), (b, c) \in R \Rightarrow |a - b|, |b - c|$ are divisible by 2

$\Rightarrow a - b = \pm 2m$ and $b - c = \pm 2n$

$\Rightarrow a - b + b - c = \pm 2(m + n)$

$\Rightarrow (a - c) = \pm 2k \quad [\because k = m + n]$

$\Rightarrow (a - c) = 2k$

$\Rightarrow (a - c)$ is divisible by 2 $\Rightarrow (a, c) \in R$.

Hence R is transitive.

Therefore, R is an equivalence relation.

The elements related to 6 are 6, 8.

26. Let * *be* a binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of *x*, given that 2 * (*x* * 5) = 10.

Ans:

Given 2 * (x * 5) = 10

$\Rightarrow 2 * \frac{x \times 5}{5} = 10 \Rightarrow 2 * x = 10$

$\Rightarrow \frac{2 \times x}{5} = 10 \Rightarrow x = \frac{10 \times 5}{5} \Rightarrow x = 25$

27. Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by (a, b) R (c, d) if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)].

Ans:

Given, R is a relation in $A \times A$ defined by $(a, b)R(c, d) \Rightarrow a + d = b + c$

(i) **Reflexivity:** $\forall a, b \in A$

Q $a + b = b + a \Rightarrow (a, b)R(a, b)$

So, R in reflexive.

(ii) **Symmetry:** Let (a, b) R (c, d)

Q $(a, b)R(c, d) \Rightarrow a + d = b + c$

$\Rightarrow b + c = d + a$ [Q $a, b, c, d \in N$ and N is commutative under addition[

$\Rightarrow c + b = d + a$

$\Rightarrow (c, d)R(a, b)$

So, R is symmetric.

(iii) **Transitivity:** Let $(a, b)R(c, d)$ and $(c, d)R(e, f)$

Now, $(a, b)R(c, d)$ and $(c, d)R(e, f) \Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow a + d + c + f = b + c + d + e$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b)R(e, f)$.

R is transitive.

Hence, R is an equivalence relation.

2nd Part: Equivalence class: $[(2, 5)] = \{(a, b) \in A \times A: (a, b)R(2, 5)\}$

$= \{(a, b) \in A \times A: a + 5 = b + 2\}$

$= \{(a, b) \in A \times A: b - a = 3\}$

$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$