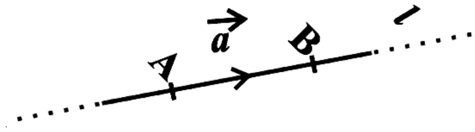
VECTOR ALGEBRA

CHAPTER – 10: VECTOR ALGEBRA

MARKS WEIGHTAGE – 06 marks

QUICK REVISION (Important Concepts & Formulae)

☞ **Vector**
The line *l* to the line segment AB, then a magnitude is prescribed on the line *l* with one of the two directions, so that we obtain a *directed line segment*. Thus, a directed line segment has magnitude as well as direction.

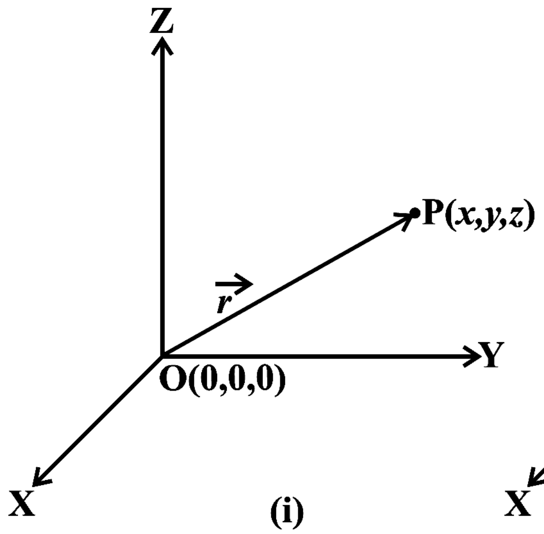


☞ A quantity that has magnitude as well as direction is called a vector.

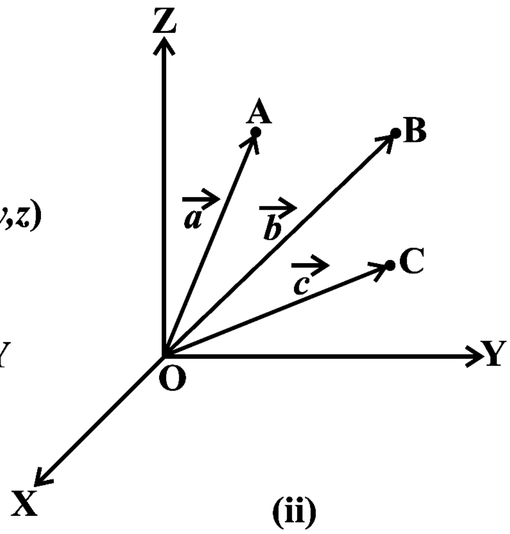
☞ A directed line segment is a vector, denoted as $\overrightarrow{AB}$ or simply as $|\vec{a}|$, and read as 'vector $\overrightarrow{AB}$' or 'vector $|\vec{a}|$'.

☞ The point A from where the vector $\overrightarrow{AB}$ starts is called its *initial point*, and the point B where it ends is called its *terminal point*. The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$. The arrow indicates the direction of the vector.

☞ The vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula, the magnitude of vector $\overrightarrow{OP}$ is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$



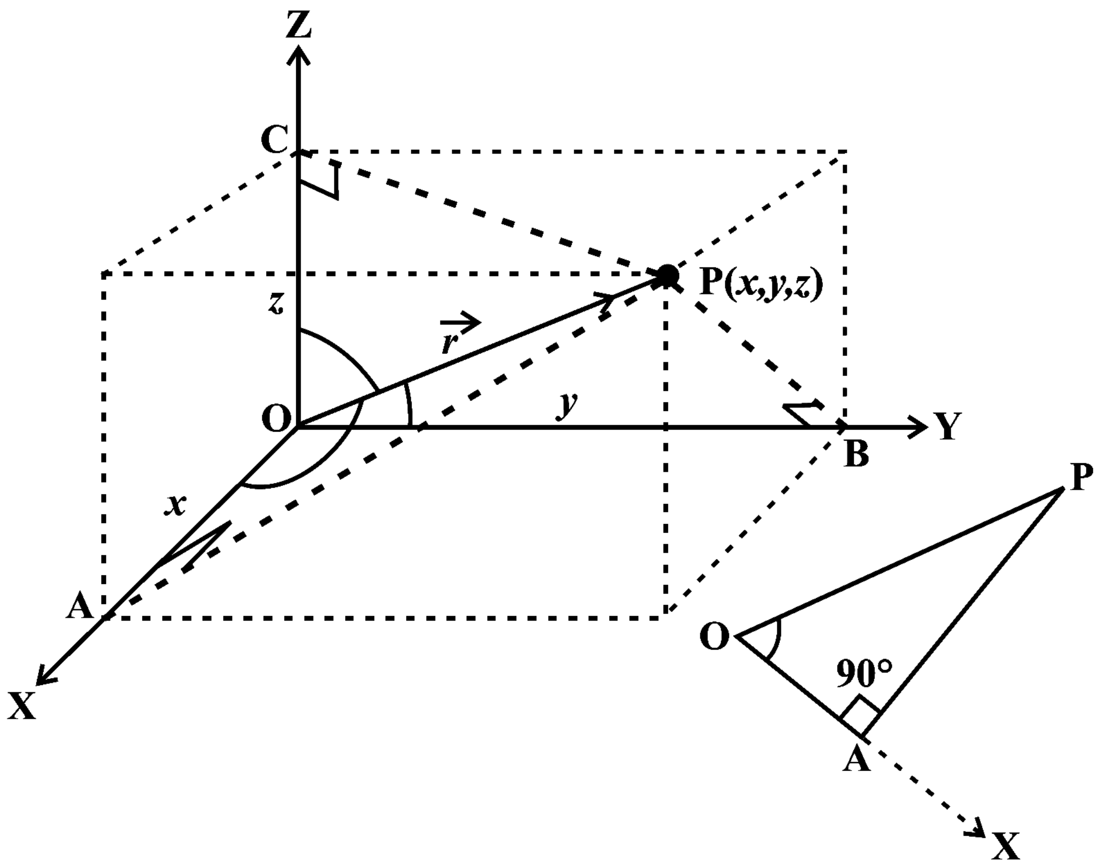
(i)



(ii)

☞ The position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$ *and* $\vec{c}$, etc., respectively

☞ ***Direction Cosines***
Consider the position vector $\overrightarrow{OP}$(*or* $\vec{r}$) of a point P(*x*, *y*, *z*) in below figure. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of *x*, *y* and *z*-axes respectively, are called its *direction angles*. The cosine values of these angles, i.e., cosα, cosβ and cos γ are called *direction cosines* of the vector $\vec{r}$, and usually denoted by *l*, *m* and *n*, respectively.



☞ The triangle OAP is right angled, and in it, we have $\cos\alpha = \frac{x}{r}$ (r stands for $|\vec{r}|$). Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus, the coordinates of the point P may also be expressed as (*lr*, *mr*, *nr*). The numbers *lr, mr* and *nr*, proportional to the direction cosines are called as *direction ratios* of vector $\vec{r}$, and denoted as *a, b* and *c,* respectively.

☞ $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

Types of Vectors

☞ **Zero Vector** A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector,

☞ **Unit Vector** A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$

☞ **Coinitial Vectors** Two or more vectors having the same initial point are called coinitial vectors.

☞ **Collinear Vectors** Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

☞ **Equal Vectors** Two vectors $\vec{a}$ *and* $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

☞ **Negative of a Vector** A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called *negative* of the given vector.
For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

☞ The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called *free vectors.*

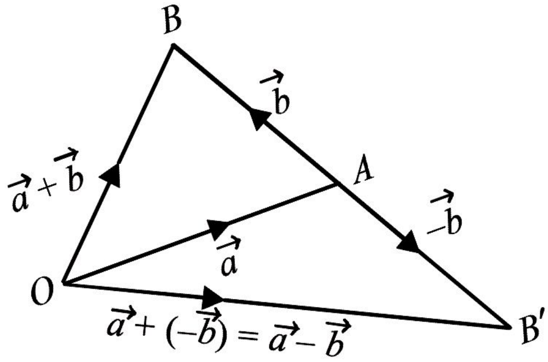
Addition of Vectors

☞ ***Triangle law of vector addition***

If two vectors $\vec{a}$ *and* $\vec{b}$ are represented (in magnitude and direction) by two sides of a triangle taken in order, then their sum (resultant) is represented by the third side $\vec{c}$ $(=\vec{a}+\vec{b})$ taken in the opposite order.

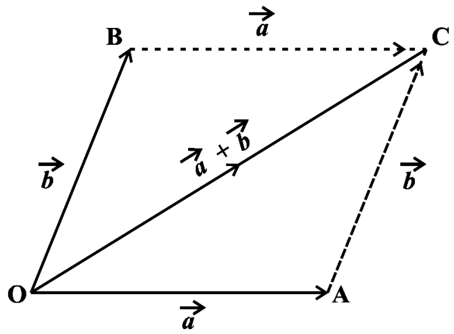
☞ **Subtraction of Vectors :** To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ and add to $\vec{a}$.

<u>Geometrical Representation of Addition and Subtraction :</u>



☞ ***Parallelogram law of vector addition***

If we have two vectors $\vec{a}$ *and* $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition.*



Properties of vector addition

☞ **Property 1**

For any two vectors $\vec{a}$ *and* $\vec{b}$, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (Commutative property)

☞ **Property 2**

For any three vectors $\vec{a},\ \vec{b}$ *and* $\vec{c}$, $\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$ (Associative property)

☞ **Property 3**

For any vector $\vec{a}$, we have $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$, Here, the zero vector $\vec{0}$ is called the *additive identity* for the vector addition.

☞ **Property 4**

For any vector $\vec{a}$, we have $\vec{a}+\left(-\vec{a}\right)=\left(-\vec{a}\right)+\vec{a}=\vec{0}$

Here, the vector $-\vec{a}$ is called the *additive inverse* for the vector addition.

☞ **Multiplication of a Vector by a Scalar**

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that, $\lambda\vec{a}$ is also a vector, collinear to the vector $\vec{a}$. The vector $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is |λ| times the magnitude of the vector $\vec{a}$, i.e., $|\lambda\vec{a}|=|\lambda||\vec{a}|$

☞ Unit vector in the direction of vector $\vec{a}$ is given by $\hat{a}=\frac{1}{|\vec{a}|}.\vec{a}$

☞ **Properties of Multiplication of Vectors by a Scalar**

For vectors $\vec{a}$, $\vec{b}$ and scalars *m*, *n*, we have

(i) $m(-\vec{a})=(-m)\vec{a}=-(m\vec{a})$

(ii) $(-m)(-\vec{a})=m\vec{a}$

(iii) $m(n\vec{a})=(mn)\vec{a}=n(m\vec{a})$

(iv) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

(v) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$.

☞ ***Vector joining two points***

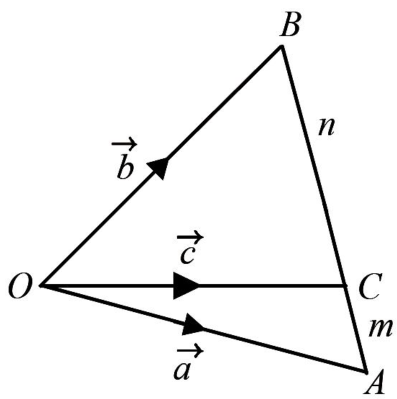
If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.

$\overrightarrow{P_1P_2}$ = Position vector of head – Position vector of tail = $\overrightarrow{OP_2}-\overrightarrow{OP_1}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

SECTION FORMULA

☞ **Internal Division :** Position vector of a point *C* dividing a vector $\overrightarrow{AB}$ internally in the ratio of *m* : *n* is $\overrightarrow{OC}=\frac{m\vec{b}+n\vec{a}}{m+n}$



☞ If *C* is the midpoint of $\overrightarrow{AB}$, then $\overrightarrow{OC}$ divides $\overrightarrow{AB}$ in the ratio 1 : 1. Therefore, position vector of *C* is $\frac{1.\vec{b}+1.\vec{a}}{1+1}=\frac{\vec{a}+\vec{b}}{2}$

☞ The position vector of the midpoint of $\overrightarrow{AB}$ is $\dfrac{\vec{a}+\vec{b}}{2}$

☞ Position vector of any point *C* on $\overrightarrow{AB}$ can always be taken as $\vec{c}=\lambda\vec{b}+\mu\vec{a}$ where $\lambda + \mu = 1$.

☞ $n.\overrightarrow{OA}+m.\overrightarrow{OB}=(n+m).\overrightarrow{OC}$, where *C* is a point on $\overrightarrow{AB}$ dividing it in the ratio *m* : *n*.

☞ **External Division :** Let *A* and *B* be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let *C* be a point dividing $\overrightarrow{AB}$ externally in the ratio *m* : *n*. Then the position vector of *C* is given by

☞ $\overrightarrow{OC} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$

☞ Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a nonzero scalar λ such that $\vec{b} = \lambda a$. If the vectors $\vec{a}$ *and* $\vec{b}$ are given in the component form, i.e. $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then the two vectors are collinear if and only if

$b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=\lambda\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$

$\Rightarrow \dfrac{b_1}{a_1}=\dfrac{b_2}{a_2}=\dfrac{b_3}{a_3}=\lambda$

☞ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

☞ In case if it is given that *l*, *m*, *n* are direction cosines of a vector, then $l\hat{i}+m\hat{j}+n\hat{k} = (\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles which the vector makes with *x*, *y* and *z* axes respectively.

Product of Two Vectors

☞ ***Scalar (or dot) product of two vectors***

The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined as

$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$

☞ If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined, and in this case, we define $\vec{a}.\vec{b}=\vec{0}$

Observations

☞ $\vec{a}.\vec{b}$ is a real number.

☞ Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a}.\vec{b}=\vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.

$\vec{a}.\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

☞ If θ = 0, then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$. In particular, $\vec{a}.\vec{a}=|\vec{a}|^2$, as θ in this case is 0.

☞ If θ = π, then $\vec{a}.\vec{b}=-|\vec{a}||\vec{b}|$ In particular, $\vec{a}.(-\vec{a})=-|\vec{a}|^2$, as θ in this case is π.

☞ In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i},\ \hat{j}$ *and* $\hat{k}$, we have

$\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$

$\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

☞ The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$, is given by

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|},\ \ or\ \ \theta=\cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

☞ The scalar product is commutative. i.e. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

- **Property 1** (Distributivity of scalar product over addition) Let $\vec{a},\ \vec{b}\ and\ \vec{c}$ be any three vectors, then $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$
- **Property 2** Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a}).\vec{b} = \lambda(\vec{a}.\vec{b}) = \vec{a}.(\lambda\vec{b})$

Projection of a vector on a line

- If $\hat{p}$ is the unit vector along a line *l*, then the projection of a vector $\vec{a}$ on the line *l* is given by $\vec{a}.\hat{p}$.
- Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by $\vec{a}.\hat{b}\ or\ \vec{a}.\left(\frac{\vec{b}}{|\vec{b}|}\right)\ or\ \frac{1}{|\vec{b}|}(\vec{a}.\vec{b})$
- If θ = 0, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if θ = π, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$
- If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.
- If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then its direction cosines may be given as $\cos\alpha = \frac{\vec{a}.\hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}, \cos\beta = \frac{a_2}{|\vec{a}|} \cos\lambda = \frac{a_3}{|\vec{a}|}$

Vector (or cross) product of two vectors

- The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a}\times\vec{b}$ and defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where, θ is the angle between $\vec{a}$ and $\vec{b}$, 0 ≤ θ ≤ π and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.
- If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a}\times\vec{b} = \vec{0}$.

Observations

- $\vec{a}\times\vec{b}$ is a vector.
- Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a}\times\vec{b} = \vec{0}$ if and only if and $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,
  $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\,||\,\vec{b}$
- In particular, $\vec{a}\times\vec{a} = \vec{0}$ and $\vec{a}\times(-\vec{a}) = \vec{0}$, since in the first situation, θ = 0 and in the second one, θ = π, making the value of sin θ to be 0.
- If $\theta = \frac{\pi}{2}$ then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|$
- In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i},\ \hat{j}\ and\ \hat{k}$, we have
  $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
  $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$
- In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as
  $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$
- The vector product is not commutative, as $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- In view of the above observations, we have $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a}\times\vec{b}|$.

☞ If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$.

☞ The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ *is* $\frac{1}{2}|\vec{a}\times\vec{b}|$

☞ $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

☞ $-\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is also a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

☞ **Property 3** (Distributivity of vector product over addition): If $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ are any three vectors and λ be a scalar, then

(i) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

(ii) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

☞ Let $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

CHAPTER – 10: VECTOR ALGEBRA

MARKS WEIGHTAGE – 06 marks

NCERT Important Questions & Answers

1. Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$

Ans:

The sum of the given vectors is $\vec{a} + \vec{b}\ \ (= \vec{c}, say) = 4\hat{i} + 3\hat{j} - 2\hat{k}$

and $|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

2. Show that the points are $A(2\hat{i} - \hat{j} + \hat{k}), B(\hat{i} - 3\hat{j} - 5\hat{k}), C(3\hat{i} - 4\hat{j} - 4\hat{k})$ the vertices of a right angled triangle.

Ans:

We have $\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

and $\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

Then $|\overrightarrow{AB}|^2 = 41, |\overrightarrow{BC}|^2 = 6, |\overrightarrow{CA}|^2 = 35$

$\Rightarrow |\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

Hence, the triangle is a right angled triangle.

3. Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1), directed from A to B.

Ans:

The given points are $A(1,2,-3)$ and $B(-1,-2,1)$.

Then $\overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + (1-(-3))\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

Now, $|\overrightarrow{AB}| = \sqrt{4+16+16} = \sqrt{36} = 6$

$\therefore$ unit vector along $\overrightarrow{AB}$ = $\frac{1}{|\overrightarrow{AB}|}\overrightarrow{AB} = \frac{1}{6}(-2\hat{i} - 4\hat{j} + 4\hat{k}) = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Hence direction cosines are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

4. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (i) internally (ii) externally

Ans:

The position vector of a point *R* divided the line segment joining two points *P* and *Q* in the ratio *m*: *n* is given by

Case I Internally $= \frac{m\vec{b} + n\vec{a}}{m+n}$

Case II Externally $= \frac{m\vec{b} - n\vec{a}}{m-n}$

Position vectors of *P* and *Q* are given as $\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}, \overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$

(i) Position vector of R [dividing (PQ) in the ratio *2 : 1* internally]

$$= \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m+n} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} - \frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(i) Position vector of R [dividing (PQ) in the ratio *2 : 1* externally]

$$= \frac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m-n} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} = \frac{-3\hat{i} + 0\hat{j} + 3\hat{k}}{1} - 3\hat{i} + 3\hat{k}$$

5. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

Ans:

Position vectors of *P* and *Q* are given as $\overrightarrow{OP} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$

The position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2) is given by

Position Vector of the mid-point of (*PQ*) = $\frac{1}{2}\left(\overrightarrow{OQ} + \overrightarrow{OP}\right) = \frac{1}{2}\left(4\hat{i} + \hat{j} - 2\hat{k} + 2\hat{i} + 3\hat{j} + 4\hat{k}\right)$

$$= \frac{1}{2}\left(6\hat{i} + 4\hat{j} + 2\hat{k}\right) = 3\hat{i} + 2\hat{j} + \hat{k}$$

6. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively form the vertices of a right angled triangle.

Ans:

Position vectors of points *A*, *B* and *C* are respectively given as

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ *and* $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

Now, $\overrightarrow{AB} = \vec{b} - \vec{a} = 2\hat{i} - \hat{j} + \hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$\Rightarrow |\overrightarrow{AB}|^2 = 1 + 9 + 25 = 35$

$\overrightarrow{BC} = \vec{c} - \vec{b} = \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\Rightarrow |\overrightarrow{BC}|^2 = 1 + 4 + 36 = 41$

$\overrightarrow{CA} = \vec{a} - \vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k} - \hat{i} + 3\hat{j} + 5\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\Rightarrow |\overrightarrow{CA}|^2 = 4 + 1 + 1 = 6$

$\Rightarrow |\overrightarrow{BC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2$

Hence it form the vertices of a right angled triangle.

7. Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Ans:

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Now, $\vec{a}.\vec{b} = (\hat{i} + \hat{j} - \hat{k}).(\hat{i} - \hat{j} + \hat{k}) = 1 - 1 - 1 = -1$

Therefore, we have $\cos\theta = \frac{-1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{3}\right)$

8. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

Ans:

We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here, $\vec{a} + \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\vec{a} - \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} - \hat{i} - 3\hat{j} + 5\hat{k} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

Now, $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = (6\hat{i}+2\hat{j}-8\hat{k}).(4\hat{i}-4\hat{j}+2\hat{k}) = 24-8-16 = 0$

Hence $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

9. Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}.\vec{b}$ = 4.

Ans:

We have

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b}).(\vec{a}-\vec{b}) = \vec{a}.\vec{a}-\vec{a}.\vec{b}-\vec{b}.\vec{a}+\vec{b}.\vec{b}$

$=|\vec{a}|^2 - 2(\vec{a}.\vec{b}) + |\vec{b}|^2 = 2^2 - 2(4) + 3^2 = 4-8+9 = 5$

$\Rightarrow |\vec{a}-\vec{b}| = \sqrt{5}$

10. Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k}), B(\hat{i}+2\hat{j}+3\hat{k}), C(7\hat{i}-\hat{k})$ are collinear.

Ans:

We have

$\overrightarrow{AB} = (1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k} = 3\hat{i}-\hat{j}-2\hat{k}$

$\overrightarrow{BC} = (7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k} = 6\hat{i}-2\hat{j}-4\hat{k}$

$\overrightarrow{CA} = (7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k} = 9\hat{i}-3\hat{j}-6\hat{k}$

Now, $|\overrightarrow{AB}|^2 = 14, |\overrightarrow{BC}|^2 = 56, |\overrightarrow{CA}|^2 = 126$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{14}, |\overrightarrow{BC}| = 2\sqrt{14}, |\overrightarrow{CA}| = 3\sqrt{14}$

$\Rightarrow |\overrightarrow{CA}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence the points A, B and C are collinear.

11. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Ans:

Given that $|\vec{a}|=1, |\vec{b}|=1, |\vec{c}|=1, \vec{a}+\vec{b}+\vec{c}=0$

$(\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = 0$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{b}.\vec{a}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c} = 0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow 1+1+1+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = -3$

$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = \frac{-3}{2}$

12. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find $\angle$ ABC.

Ans:

We are given the points $A(1, 2, 3)$, $B(-1, 0, 0)$ and $C(0, 1, 2)$.

Also, it is given that $\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$

Now, $\overrightarrow{BA} = (\hat{i}+2\hat{j}+3\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k}) = 2\hat{i}+2\hat{j}+3\hat{k}$

$\Rightarrow |\overrightarrow{BA}| = \sqrt{4+4+9} = \sqrt{17}$

and $\overrightarrow{BC} = (0\hat{i}+\hat{j}+2\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k}) = \hat{i}+\hat{j}+2\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{1+1+4} = \sqrt{6}$

$\overrightarrow{BA}.\overrightarrow{BC} = (2\hat{i}+2\hat{j}+3\hat{k}).(\hat{i}+\hat{j}+2\hat{k}) = 2+2+6 = 10$

$\cos\theta = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \Rightarrow \cos\angle ABC == \frac{10}{(\sqrt{17})(\sqrt{6})} = \frac{10}{\sqrt{102}}$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

13. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

Ans:

The given points are $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$.

$\overrightarrow{AB} = (2\hat{i}+6\hat{j}+3\hat{k})-(\hat{i}+2\hat{j}+7\hat{k}) = \hat{i}+4\hat{j}-4\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1+16+16} = \sqrt{33}$

$\overrightarrow{BC} = (3\hat{i}+10\hat{j}-\hat{k})-(2\hat{i}+6\hat{j}+3\hat{k}) = \hat{i}+4\hat{j}-4\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{1+16+16} = \sqrt{33}$

and $\overrightarrow{AC} = (3\hat{i}+10\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+7\hat{k}) = 2\hat{i}+8\hat{j}-8\hat{k}$

$\Rightarrow |\overrightarrow{AC}| = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence, the given points A, B and C are collinear.

14. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Ans:

Let A= $2\hat{i}-\hat{j}+\hat{k}$ B = $\hat{i}-3\hat{j}-5\hat{k}$ and C = $3\hat{i}-4\hat{j}-4\hat{k}$

$\overrightarrow{AB} = (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1+4+36} = \sqrt{41}$

$\overrightarrow{BC} = (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{4+1+1} = \sqrt{6}$

and $\overrightarrow{AC} = (3\hat{i}-4\hat{j}-4\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = \hat{i}-3\hat{j}-5\hat{k}$

$\Rightarrow |\overrightarrow{AC}| = \sqrt{1+9+25} = \sqrt{35}$

$\therefore |\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2$

Hence, ABC is a right angled triangle.

15. Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a} = \hat{i}+\hat{j}+\hat{k}, \vec{b} = \hat{i}+2\hat{j}+3\hat{k}$.

Ans:

We have $\vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b} = -\hat{j}-2\hat{k}$

A vector which is perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} (= \vec{c}, say)$$

Now, $|\vec{c}| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

Therefore, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{2\sqrt{6}}(-2\hat{i}+4\hat{j}-2\hat{k}) = \frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\hat{k}$$

16. Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Ans:

We have $\overrightarrow{AB} = \hat{j}+2\hat{k}$ and $\overrightarrow{AC} = \hat{i}+2\hat{j}$.

The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=-4\hat{i}+2\hat{j}-\hat{k}$

Therefore, $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{16+4+1}=\sqrt{21}$

Thus, the required area is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{21}$

17. Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

Ans:

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 4\\ 1 & -1 & 1\end{vmatrix}=5\hat{i}+\hat{j}-4\hat{k}$

Therefore, $|\vec{a}\times\vec{b}|=\sqrt{25+1+16}=\sqrt{42}$

and hence, the required area is $\sqrt{42}$.

18. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Ans:

$\overrightarrow{AB}=(2\hat{i}+3\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k})=\hat{i}+2\hat{j}+3\hat{k}$

$\overrightarrow{AC}=(\hat{i}+5\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k})=4\hat{j}+3\hat{k}$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}=-6\hat{i}-3\hat{j}+4\hat{k}$

$\Rightarrow|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{36+9+16}=\sqrt{61}$

Area of triangle ABC = $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{\sqrt{61}}{2}$ sq. units.

19. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Ans:

Adjacent sides of parallelogram are given by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=20\hat{i}+5\hat{j}-5\hat{k}$

$\Rightarrow|\vec{a}\times\vec{b}|=\sqrt{400+25+25}=\sqrt{450}=15\sqrt{2}$

Hence, the area of the given parallelogram is $15\sqrt{2}$ sq. units.

20. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

Given that vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

Also, $\vec{a}\times\vec{b}$ is a unit vector $\Rightarrow |\vec{a}\times\vec{b}| = 1$

$\Rightarrow |\vec{a}|.|\vec{b}|\sin\theta = 1 \Rightarrow 3\times\frac{\sqrt{2}}{3}\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

21. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overline{AB}$ and $\overline{CD}$. Deduce that $\overline{AB}$ and $\overline{CD}$ are collinear.

Ans:

Note that if θ is the angle between $\overline{AB}$ and $\overline{CD}$, then θ is also the angle between $\overline{AB}$ and $\overline{CD}$.

Now $\overline{AB}$ = Position vector of B – Position vector of A

$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$

Therefore, $|\overrightarrow{AB}| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

Similarly, $\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k} \Rightarrow |\overrightarrow{CD}| = \sqrt{4+64+4} = \sqrt{72} = 6\sqrt{2}$

Thus, $\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = -1$

Since 0 ≤ θ ≤ π, it follows that θ = π. This shows that $\overline{AB}$ and $\overline{CD}$ are collinear.

22. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Ans:

Given that each one of them being perpendicular to the sum of the other two.

Therefore, $\vec{a}.(\vec{b}+\vec{c}) = 0, \vec{b}.(\vec{c}+\vec{a}) = 0, \vec{c}.(\vec{a}+\vec{b}) = 0$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$= \vec{a}.\vec{a}+\vec{a}.(\vec{b}+\vec{c})+\vec{b}.\vec{b}+\vec{b}.(\vec{c}+\vec{a})+\vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$

$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$

$= 9+16+25 = 50$

Therefore, $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

23. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$.

Ans:

Given vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$.

Let $\vec{c}$ be the resultant vector $\vec{a}$ and $\vec{b}$ then

$\vec{c} = (2\hat{i}+3\hat{j}-\hat{k})+(\hat{i}-2\hat{j}+\hat{k}) = 3\hat{i}+\hat{j}+0\hat{k}$

$\Rightarrow |\vec{c}| = \sqrt{9+1+0} = \sqrt{10}$

$\therefore$ Unit vector in the direction of $\vec{c}$ = $\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$\pm 5\hat{c} = \pm 5\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j}) = \pm\frac{3\sqrt{10}}{2}\hat{i} \pm \frac{\sqrt{10}}{2}\hat{j}$

24. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Ans:

Two adjacent sides of a parallelogram are given by $\vec{a}=2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$

Then the diagonal of a parallelogram is given by $\vec{c}=\vec{a}+\vec{b}$

$\therefore\ \vec{c}=\vec{a}+\vec{b}=2\hat{i}-4\hat{j}+5\hat{k}+\hat{i}-2\hat{j}-3\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$

$\Rightarrow|\vec{c}|=\sqrt{9+36+4}=\sqrt{49}=7$

Unit vector parallel to its diagonal = $\hat{c}=\frac{1}{|\vec{c}|}\vec{c}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}=22\hat{i}+11\hat{j}+0\hat{k}$

Then the area of a parallelogram = $|\vec{a}\times\vec{b}|=\sqrt{484+121+0}=\sqrt{605}=11\sqrt{5}$ sq. units.

25. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$.

Ans:

The vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ must be parallel to $\vec{a}\times\vec{b}$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix}=32\hat{i}-\hat{j}-14\hat{k}$

Let $\vec{d}=\lambda(\vec{a}\times\vec{b})=\lambda(32\hat{i}-\hat{j}-14\hat{k})$

Also $\vec{c}.\vec{d}=15\Rightarrow(2\hat{i}-\hat{j}+4\hat{k}).\lambda(32\hat{i}-\hat{j}-14\hat{k})=15$

$\Rightarrow 64\lambda+\lambda-56\lambda=15\Rightarrow 9\lambda=15\Rightarrow\lambda=\frac{15}{9}=\frac{5}{3}$

$\therefore$ Required vector $\vec{d}=\frac{5}{3}(32\hat{i}-\hat{j}-14\hat{k})$

26. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Ans: Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Now, $\vec{b}+\vec{c}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow|\vec{b}+\vec{c}|=\sqrt{(2+\lambda)^2+36+4}=\sqrt{4+\lambda^2+4\lambda+40}=\sqrt{\lambda^2+4\lambda+44}$

Unit vector along $\vec{b}+\vec{c}$ is $\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$

The scalar product of $\hat{i}+\hat{j}+\hat{k}$ with this unit vector is 1.

$\therefore(\hat{i}+\hat{j}+\hat{k}).\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=1\Rightarrow(\hat{i}+\hat{j}+\hat{k}).\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1\Rightarrow\frac{\lambda+6}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\lambda+6=\sqrt{\lambda^2+4\lambda+44}\Rightarrow(\lambda+6)^2=\lambda^2+4\lambda+44$

$\Rightarrow\lambda^2+12\lambda+36=\lambda^2+4\lambda+44\Rightarrow 8\lambda=8\Rightarrow\lambda=1$

27. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Ans:

Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

$\therefore\ \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

It is also given that $|\vec{a}| = |\vec{b}| = |\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$ at angles α, β and γ respectively.

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\beta = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|} = \frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{0+|\vec{b}|^2+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\gamma = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} = \frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{0+0+|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, therefore, cosα = cosβ = cosγ

$\therefore\ \alpha = \beta = \gamma$

Hence, the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

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CHAPTER – 10: VECTOR ALGEBRA

MARKS WEIGHTAGE – 06 marks

Previous Years Board Exam (Important Questions & Answers)

1. **Write the projection of vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$.**

   Ans:

   Required projection $=\dfrac{(\hat{i}+\hat{j}+\hat{k}).\hat{j}}{|\hat{j}|}=\dfrac{0+1+0}{\sqrt{0+1+0}}=\dfrac{1}{1}=1$

2. **Find a vector in the direction of vector $2\hat{i}-3\hat{j}+6\hat{k}$ which has magnitude 21 units.**

   Ans:

   Required vector $=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}\right)=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{49}}\right)$

   $=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{7}\right)=3(2\hat{i}-3\hat{j}+6\hat{k})=6\hat{i}-9\hat{j}+18\hat{k}$

3. **Show that the vectors $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar if $\vec{a},\vec{b},\vec{c}$ are coplanar.**

   Ans:

   Let $\vec{a},\vec{b},\vec{c}$ are coplanar then we have $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

   $\Rightarrow \vec{a}.(\vec{b}\times\vec{c})=\vec{b}.(\vec{c}\times\vec{a})=\vec{c}.(\vec{a}\times\vec{b})=0$

   Now, $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=(\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

   $=(\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}$

   $=(\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}$

   $=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

   $=\left[\vec{a}\ \vec{b}\ \vec{c}\right]+0+0+0+0+\left[\vec{b}\ \vec{c}\ \vec{a}\right]$

   $=\left[\vec{a}\ \vec{b}\ \vec{c}\right]+\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

   $=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]=2\times0=0$

   Hence, $\vec{a},\vec{b},\vec{c}$ are coplanar

4. **Show that the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.**

   Ans:

   Let $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are coplanar

   $\Rightarrow (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}=0$

   $\Rightarrow (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}=0$

   $\Rightarrow (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}=0$

   $\Rightarrow \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})=0$

   $\Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]+0+0+0+0=0$

   $\Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

   $\Rightarrow \vec{a},\vec{b},\vec{c}$ are coplanar

5. **Write a unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 0) and (4, 5, 6) respectively.**

Ans:

$\overrightarrow{PQ} = (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore$ Required unit vector = $\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{9+4+36}} = \dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{49}} = \dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{7} = \dfrac{3}{7}\hat{i} + \dfrac{2}{7}\hat{j} + \dfrac{6}{7}\hat{k}$

6. **Write the value of the following :** $\hat{i}\times(\hat{j}+\hat{k}) + \hat{j}\times(\hat{k}+\hat{i}) + \hat{k}\times(\hat{i}+\hat{j})$

Ans:

$\hat{i}\times(\hat{j}+\hat{k}) + \hat{j}\times(\hat{k}+\hat{i}) + \hat{k}\times(\hat{i}+\hat{j})$

$= \hat{i}\times\hat{j} + \hat{i}\times\hat{k} + \hat{j}\times\hat{k} + \hat{j}\times\hat{i} + \hat{k}\times\hat{i} + \hat{k}\times\hat{j}$

$= \hat{k} - \hat{j} + \hat{i} - \hat{k} + \hat{j} - \hat{i} = 0$

7. **Find the value of 'p' for which the vectors** $3\hat{i}+2\hat{j}+9\hat{k}$ **and** $\hat{i}-2p\hat{j}+3\hat{k}$ **are parallel.**

Ans:

Since given two vectors are parallel.

$\Rightarrow \dfrac{3}{1} = \dfrac{2}{-2p} = \dfrac{9}{3} \Rightarrow \dfrac{3}{1} = \dfrac{2}{-2p}$

$\Rightarrow -6p = 2 \Rightarrow p = -\dfrac{1}{3}$

8. **Find** $\vec{a}.(\vec{b}\times\vec{c})$**, if** $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}, \vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ **and** $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$**.**

Ans:

Given that $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}, \vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$

$$\therefore \vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2(4-1) - 1(-2-3) + 3(-1-6)$$

$= 6 + 5 - 21 = -10$

9. **Show that the four points *A, B, C* and *D* with position vectors** $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ **and** $4(-\hat{i}+\hat{j}+\hat{k})$ **are coplanar.**

Ans:

Position vectors of *A*, *B*, *C* and *D* are

Position vector of *A* = $4\hat{i}+5\hat{j}+\hat{k}$

Position vector of *B* = $-\hat{j}-\hat{k}$

Position vector of *C* = $3\hat{i}+9\hat{j}+4\hat{k}$

Position vector of *D* = $4(-\hat{i}+\hat{j}+\hat{k}) = -4\hat{i}+4\hat{j}+4\hat{k}$

$\therefore \overrightarrow{AB} = -4\hat{i}-6\hat{j}-2\hat{k}, \overrightarrow{AC} = -\hat{i}+4\hat{j}+3\hat{k}, \overrightarrow{AD} = -8\hat{i}-\hat{j}+3\hat{k}$

$$\text{Now, } \overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3) + 6(-3+24) - 2(1+32)$$

$= -60 + 126 - 66 = 0$

$\Rightarrow \overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = 0$

Hence $\overrightarrow{AB}$, $\overrightarrow{AC}$ *and* $\overrightarrow{AD}$ are coplanar i.e. A, B, C and D are coplanar.

10. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with *x*-axis., $\frac{\pi}{2}$ with y-axis and an acute angle θ with *z*-axis.

Ans:

Direction cosines of required vector $\vec{a}$ are

$l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{2} = 0$ and $n = \cos\theta$

$\because l^2 + m^2 + n^2 = 1$

$\therefore \left(\frac{1}{\sqrt{2}}\right)^2 + 0 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow n = \frac{1}{\sqrt{2}}$

$\therefore$ Unit vector in the direction of $\vec{a} = \frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

$\therefore \vec{a} = 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5\hat{i} + 5\hat{k}$

11. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|$ = 13 and $|\vec{a}|$ = 5 find the value of $|\vec{b}|$.

Ans:

Given $|\vec{a}+\vec{b}|$ = 13

$|\vec{a}+\vec{b}|^2 = 169 \Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 169$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 169$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 169 \quad \left[\because \vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0\right]$

$\Rightarrow |\vec{b}|^2 = 169 - |\vec{a}|^2 = 169 - 25 = 144$

$\Rightarrow |\vec{b}| = 12$

12. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Ans:

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Projection of the vector $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{(\hat{i} + 3\hat{j} + 7\hat{k}).(2\hat{i} - 3\hat{j} + 6\hat{k})}{|2\hat{i} - 3\hat{j} + 6\hat{k}|}$

$= \frac{2 - 9 + 42}{\sqrt{4+9+36}} = \frac{35}{\sqrt{49}} = \frac{35}{7} = 5$

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

Given that $\vec{a}+\vec{b}$ is also a unit vector

$\therefore |\vec{a}+\vec{b}| = 1$

$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1^2 = 1$

$\Rightarrow 1 + 2\vec{a}.\vec{b} + 1 = 1 \quad \left[\because |\vec{a}| = 1, |\vec{b}| = 1\right]$

$\Rightarrow 2\vec{a}.\vec{b} = -1 \Rightarrow \vec{a}.\vec{b} = -\frac{1}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2}$

$$\Rightarrow 1\times 1\times \cos\theta = -\frac{1}{2} \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \cos\theta = \cos\frac{2\pi}{3}$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

14. Prove that, for any three vectors $\vec{a},\ \vec{b},\ \vec{c}$

$$\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$$

Ans:

$$\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}$$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+0+0+0+0+\left[\vec{b}\ \ \vec{c}\ \ \vec{a}\right]$$

$$= \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$$

$$= 2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$$

15. Vectors $\vec{a},\ \vec{b},\ \vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}$ = 0 and $|\vec{a}|=3,\ |\vec{b}|=5\ \ and\ \ |\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

$$\vec{a}+\vec{b}+\vec{c}=0 \Rightarrow \vec{a}+\vec{b}=-\vec{c}$$

$$\Rightarrow (\vec{a}+\vec{b})^2 = (-\vec{c})^2$$

$$|\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = \vec{c}.\vec{c}$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{c}|^2 \Rightarrow 9 + 2\vec{a}.\vec{b} + 25 = 49$$

$$\Rightarrow 2\vec{a}.\vec{b} = 49-25-9 = 15$$

$$\Rightarrow \vec{a}.\vec{b} = \frac{15}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = \frac{15}{2}$$

$$\Rightarrow 3\times 5\times\cos\theta = \frac{15}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

16. If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})(\vec{x}+\vec{a}) = 24$, then write the value of $|\vec{x}|$.

Ans:

Given that $(\vec{x}-\vec{a})(\vec{x}+\vec{a}) = 24$

$$\Rightarrow \vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a} = 24$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 24 \qquad \left[\because \vec{x}.\vec{a} = \vec{a}.\vec{x}\right]$$

$$\Rightarrow |\vec{x}|^2 - 1 = 24 \Rightarrow |\vec{x}|^2 = 25 \Rightarrow |\vec{x}| = 5$$

17. For any three vectors $\vec{a},\ \vec{b}\ \ and\ \ \vec{c}$, write the value of the following:

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$$

Ans:

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$$

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$$

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c} = 0$$

18. The magnitude of the vector product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

Ans:

Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$

Now, $\vec{b}+\vec{c} = 2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow |\vec{b}+\vec{c}| = \sqrt{(2+\lambda)^2+36+4} = \sqrt{4+\lambda^2+4\lambda+40} = \sqrt{\lambda^2+4\lambda+44}$

The vector product of $\hat{i}+\hat{j}+\hat{k}$ with this unit vector is $\sqrt{2}$.

$\therefore \left|\vec{a}\times\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}\right| = \sqrt{2} \Rightarrow \left|\frac{\vec{a}\times(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\right| = \sqrt{2}$

Now, $\vec{a}\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2+\lambda & 6 & -2 \end{vmatrix} = (-2-6)\hat{i}-(-2-2-\lambda)\hat{j}-(6-2-\lambda)\hat{k}$

$= -8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}$

$\therefore \left|\frac{\vec{a}\times(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\right| = \sqrt{2} \Rightarrow \left|\frac{-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right| = \sqrt{2}$

$\Rightarrow \frac{\sqrt{64+(4+\lambda)^2+(4-\lambda)^2}}{\sqrt{\lambda^2+4\lambda+44}} = \sqrt{2}$

$\frac{64+(4+\lambda)^2+(4-\lambda)^2}{\lambda^2+4\lambda+44} = 2 \Rightarrow \frac{64+16+\lambda^2+8\lambda+16+\lambda^2-8\lambda}{\lambda^2+4\lambda+44} = 2 \Rightarrow \frac{96+2\lambda^2}{\lambda^2+4\lambda+44} = 2$

$\Rightarrow 96+2\lambda^2 = 2(\lambda^2+4\lambda+44) \Rightarrow 96+2\lambda^2 = 2\lambda^2+8\lambda+88$

$\Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$

19. Find a unit vector perpendicular to each of the vectors $\vec{a}+2\vec{b}$ and $2\vec{a}+\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.

Ans.

Given that $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

$\therefore \vec{a}+2\vec{b} = 3\hat{i}+2\hat{j}+2\hat{k}+2(\hat{i}+2\hat{j}-2\hat{k})$

$= 3\hat{i}+2\hat{j}+2\hat{k}+2\hat{i}+4\hat{j}-4\hat{k} = 5\hat{i}+6\hat{j}-2\hat{k}$

and $2\vec{a}+\vec{b} = 2(3\hat{i}+2\hat{j}+2\hat{k})+\hat{i}+2\hat{j}-2\hat{k}$

$= 6\hat{i}+4\hat{j}+4\hat{k}+\hat{i}+2\hat{j}-2\hat{k} = 7\hat{i}+6\hat{j}+2\hat{k}$

Now, perpendicular vector of $\vec{a}+2\vec{b}$ and $2\vec{a}+\vec{b}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = (12+12)\hat{i}-(10+14)\hat{j}+(30-42)\hat{k}$

$= 24\hat{i}-24\hat{j}-12\hat{k} = 12(2\hat{i}-2\hat{j}-\hat{k})$

$\therefore$ Required unit vector $= \pm\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{12\sqrt{4+4+1}} = \pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{9}}$

$= \pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3} = \pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right)$

20. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

Ans:

Given that $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\therefore\ \vec{a} + \vec{b} = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k} = 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$

and $\vec{a} - \vec{b} = \hat{i} - \hat{j} + 7\hat{k} - 5\hat{i} + \hat{j} - \lambda\hat{k} = -4\hat{i} + (7-\lambda)\hat{k}$

Now, $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

$\Rightarrow (6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}).(-4\hat{i} + (7-\lambda)\hat{k} = 0$

$\Rightarrow -24 + 0 + (7+\lambda)(7-\lambda) = 0$

$\Rightarrow -24 + 49 - \lambda^2 = 0 \Rightarrow \lambda^2 = 25 \Rightarrow \lambda = \pm 5$