INVERSE TRIGONOMETRIC FUNCTIONS

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 05 marks

QUICK REVISION (Important Concepts & Formulae)

Inverse Trigonometrical Functions

A function $f : A \to B$ is invertible if it is a bijection. The inverse of f is denoted by f^{-1} and is defined as $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

- Clearly, domain of f^{-1} = range of f and range of f^{-1} = domain of f.
- The inverse of sine function is defined as $\sin^{-1}x = \theta \Leftrightarrow$ sinq $= x$, where $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$.
- Thus, sin –1 x has infinitely many values for given $x \in [-1, 1]$
- There is one value among these values which lies in the interval $[-\pi/2, \pi/2]$. This value is called the principal value.

Domain and Range of Inverse Trigonometrical Functions

Function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

Properties of Inverse Trigonometrical Functions

- $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1}x) = x$, provided that $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1}x) = x$, provided that $-1 \le x \le 1$ and $0 \le \theta \le \pi$
- $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1}x) = x$, provided that $-\infty \le x \le \infty$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1}x) = x$, provided that $-\infty < x < \infty$ and $0 < \theta < \pi$.
- $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1}x) = x$
- $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$ and $\text{cosec}(\text{cosec}^{-1}x) = x$,
- $\sin^{-1}x = \cos ec^{-1}\frac{1}{x}$ *or* $\cos ec^{-1}x = \sin^{-1}\frac{1}{x}$

- ☞ $\cos^{-1}x = \sec^{-1}\frac{1}{x}$ *or* $\sec^{-1}x = \cos^{-1}\frac{1}{x}$

- ☞ $\tan^{-1}x = \cot^{-1}\frac{1}{x}$ *or* $\cot^{-1}x = \tan^{-1}\frac{1}{x}$

- ☞ $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$

- ☞ $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x}$

- ☞ $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$

- ☞ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \textit{where} -1 \le x \le 1$

- ☞ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \textit{where} -\infty \le x \le \infty$

- ☞ $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}, \textit{where}\ \ x \le -1\ \ \textit{or}\ \ x \ge 1$

- ☞ $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \textit{if}\ \ xy < 1$

- ☞ $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \textit{if}\ \ xy > 1$

- ☞ $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

- ☞ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), \textit{if}\ \ x, y \ge 0, x^2 + y^2 \le 1$

- ☞ $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), \textit{if}\ \ x, y \ge 0, x^2 + y^2 \le 1$

- ☞ $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), \textit{if}\ \ x, y \ge 0, x^2 + y^2 > 1$

- ☞ $\sin^{-1}x - \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), \textit{if}\ \ x, y \ge 0, x^2 + y^2 > 1$

- ☞ $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), \textit{if}\ \ x, y \ge 0, x^2 + y^2 \le 1$

☞ $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 \leq 1$

☞ $\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 > 1$

☞ $\cos^{-1}x - \cos^{-1}y = \pi - \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 > 1$

☞ $\sin^{-1}(-x) = -\sin^{-1}x, \quad \cos^{-1}(-x) = \pi - \cos^{-1}x$

☞ $\tan^{-1}(-x) = -\tan^{-1}x, \quad \cot^{-1}(-x) = \pi - \cot^{-1}x$

☞ $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \qquad 2\cos^{-1}x = \cos^{-1}\left(2x^2 - 1\right)$

☞ $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

☞ $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right), \quad 3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right)$

☞ $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. **Find the principal values of** $\sin^{-1}\left(-\frac{1}{2}\right)$**.**

   Ans:

   Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta \Rightarrow \sin\theta=-\frac{1}{2}$

   We know that the range of principal value of $\sin^{-1}\theta\ \ is\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

   $\therefore \sin\theta=-\frac{1}{2}=\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right) \qquad (\because \sin(-\theta)=-\sin\theta)$

   $\Rightarrow \theta=-\frac{\pi}{6}, where\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\Rightarrow \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

   Hence the principal value of $\sin^{-1}\left(-\frac{1}{2}\right) is -\frac{\pi}{6}$

2. **Find the principal values of** $\tan^{-1}\left(-\sqrt{3}\right)$

   Ans:

   Let $\tan^{-1}\left(-\sqrt{3}\right)=\theta \Rightarrow \tan\theta=-\sqrt{3}$

   We know that the range of principal value of $\tan^{-1}\theta\ \ is\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

   $\therefore \tan\theta=-\sqrt{3}=-\tan\frac{\pi}{3}=\tan\left(-\frac{\pi}{3}\right) \qquad (\because \tan(-\theta)=-\tan\theta)$

   $\Rightarrow \theta=-\frac{\pi}{3}, where\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\Rightarrow \tan^{-1}\left(-\sqrt{3}\right)=-\frac{\pi}{3}$

   Hence the principal value of $\tan^{-1}\left(-\sqrt{3}\right) is -\frac{\pi}{3}$

3. **Find the principal values of** $\cos^{-1}\left(-\frac{1}{2}\right)$

   Ans:

   Let $\cos^{-1}\left(-\frac{1}{2}\right)=\theta \Rightarrow \cos\theta=-\frac{1}{2}$

   We know that the range of principal value of $\cos^{-1}\theta\ \ is\ [0,\pi]$

   $\therefore \cos\theta=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)$

   $=\cos\frac{2\pi}{3} \qquad (\because \cos(\pi-\theta)=-\cos\theta)$

   $\Rightarrow \theta=\frac{2\pi}{3}, where\ \ \theta\in[0,\pi]\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

   Hence the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)\ is\ \frac{2\pi}{3}$

4. Find the principal values of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Ans:

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$

We know that the range of principal value of $\cos^{-1}\theta$ *is* $[0,\pi]$

$\therefore \cos\theta = -\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \qquad (\because \cos(\pi-\theta) = -\cos\theta)$

$\Rightarrow \theta = \frac{3\pi}{4}, where\ \ \theta \in [0,\pi] \Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \frac{3\pi}{4}$

Hence the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ *is* $\frac{3\pi}{4}$

5. Find the principal values of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Ans:

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = \theta \Rightarrow \operatorname{cosec}\theta = -\sqrt{2}$

We know that the range of principal value of $\operatorname{cosec}^{-1}\theta$ *is* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$

$\therefore \operatorname{cosec}\theta = -\sqrt{2} = -\operatorname{cosec}\frac{\pi}{4} = \operatorname{cosec}\left(-\frac{\pi}{4}\right) \qquad (\because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta)$

$\Rightarrow \theta = -\frac{\pi}{4}, where\ \ \theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\} \Rightarrow \operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

Hence the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) is -\frac{\pi}{4}$

6. Find the values of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Ans:

Let $\tan^{-1}(1) = x \Rightarrow \tan x = 1 = \tan\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}\ where\ \ x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y \Rightarrow \cos y = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \qquad (\because \cos(\pi-\theta) = -\cos\theta)$

$\Rightarrow y = \frac{2\pi}{3}\ where\ \ y \in [0,\pi]$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z \Rightarrow \sin z = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right) \Rightarrow z = -\frac{\pi}{6}\ where\ \ z \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = x + y + z = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

7. Prove that $3\sin^{-1}x = \sin^{-1}(3x-4x^3), x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

Ans:

$Let \ \sin^{-1}x = \theta \Rightarrow x = \sin\theta, \ then$

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore 3\theta = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(3x-4x^3)$

$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

8. Prove that $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Ans:

$Given \ \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$LHS = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{48+77}{264}}{1-\frac{14}{264}}\right) = \tan^{-1}\left(\frac{\frac{125}{264}}{\frac{264-14}{264}}\right) = \tan^{-1}\left(\frac{\frac{125}{264}}{\frac{250}{264}}\right) = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = RHS$$

9. Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Ans:

$Given \ 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$LHS = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right) + \tan^{-1}\frac{1}{7} \qquad \left(\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

$$= \tan^{-1}\frac{1}{1-\frac{1}{4}} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{1-\frac{4}{21}}\right) = \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right) = \tan^{-1}\frac{31}{17} = RHS$$

10. Simplify : $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Ans:

Let $x = \tan\theta$, then $\theta = \tan^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{2} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \tan^{-1}\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\sec\theta - 1}{\tan\theta} = \tan^{-1}\left(\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \quad \left[\begin{array}{l} \because 1-\cos\theta = 2\sin^2\frac{\theta}{2} \\ and \;\; \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{array}\right]$$

$$= \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \quad \text{[using (i)]}$$

11. Simplify : $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Ans:

Let $x = \sec\theta$, then $\theta = \sec^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}} = \tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \tan^{-1}\frac{1}{\tan\theta} = \tan^{-1}(\cot\theta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)\left(\because \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta\right)$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1}x \quad \text{[using (i)]}$$

12. Simplify : $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Ans:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right)$$

(inside the bracket divide numerator and denominator by cos x)

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) \quad \left(\because \tan\left(\frac{\pi}{4} - x\right) = \frac{1-\tan x}{1+\tan x}\right)$$

$$= \frac{\pi}{4} - x$$

13. Simplify : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \;\; and \;\; xy < 1$

Ans:

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \;\; and \;\; xy < 1$$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} \;\; and \;\; 2\tan^{-1}y = \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \tan\frac{1}{2}\left[(2\tan^{-1}x + 2\tan^{-1}y)\right] = \tan\left[\frac{1}{2}.2(\tan^{-1}x + \tan^{-1}y)\right] = \tan(\tan^{-1}x + \tan^{-1}y)$$

$$= \tan\left(\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right) \quad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \frac{x+y}{1-x.y}$$

14. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$**, find the value of x.**

Ans:

Given that $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\frac{\pi}{4} \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{1-\left(\frac{x^2-1}{x^2-4}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{\frac{(x^2+2x-x-2)+(x^2-2x+x-2)}{x^2-4}}{\left(\frac{x^2-4-x^2+1}{x^2-4}\right)}\right)=\tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{2x^2-4}{-3}\right)=1 \Rightarrow\ 2x^2-4=-3 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

15. Find the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$**.**

Ans:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi-\frac{5\pi}{6}\right)\right) where, \frac{5\pi}{6}\in[0,\pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)=\frac{5\pi}{6}\ \left(\because \cos(2\pi-\theta)=\cos\theta\right)$$

16. Prove that $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Ans:

Given $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Let $\cos^{-1}\frac{12}{13}=x \Rightarrow \cos x=\frac{12}{13}$

$$\therefore \sin x=\sqrt{1-\cos^2 x}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$

$$\Rightarrow x=\sin^{-1}\frac{5}{13}$$

$$LHS=\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right) \qquad \left[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right]$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{16}{25}}+\frac{3}{5}\sqrt{\frac{144}{169}}\right)=\sin^{-1}\left(\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right)$$

$$= \sin^{-1}\left(\frac{20}{65}+\frac{36}{65}\right)=\sin^{-1}\frac{56}{65}=RHS$$

17. Prove that $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Ans:

$$RHS=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$$

$$Let\ \sin^{-1}\frac{5}{13}=x\Rightarrow \sin x=\frac{5}{13}$$

$$\therefore \cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\Rightarrow \tan x=\frac{\sin x}{\cos x}=\frac{\frac{5}{13}}{\frac{12}{13}}=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$$

$$Let\ \cos^{-1}\frac{3}{5}=y\Rightarrow \cos y=\frac{3}{5}$$

$$\therefore \sin y=\sqrt{1-\cos^2 y}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\Rightarrow \tan y=\frac{\sin x}{\cos x}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$$

then the equation becomes $\tan^{-1}\frac{63}{16}=x+y$

$$\Rightarrow \tan^{-1}\frac{63}{16}=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}$$

$$RHS=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}}\right) \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\left(\frac{\frac{15+48}{36}}{1-\frac{20}{36}}\right)=\tan^{-1}\left(\frac{\frac{63}{36}}{\frac{16}{36}}\right)=\tan^{-1}\left(\frac{63}{16}\right)=LHS$$

18. Prove that $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Ans:

$$LHS=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}.\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}.\frac{1}{8}}\right) \qquad \left(\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{7+5}{35}}{1-\frac{1}{35}}\right)+\tan^{-1}\left(\frac{\frac{8+3}{24}}{1-\frac{1}{24}}\right) = \tan^{-1}\left(\frac{\frac{12}{35}}{\frac{30}{35}}\right)+\tan^{-1}\left(\frac{\frac{11}{24}}{\frac{23}{24}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}.\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{391}}{1-\frac{66}{391}}\right) = \tan^{-1}\left(\frac{\frac{325}{391}}{\frac{325}{391}}\right) = \tan^{-1}(1) = \frac{\pi}{4} = RHS$$

19. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

Ans:

Given $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

$$LHS = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right) \text{ (by rationalizing the denominator)}$$

$$= \cot^{-1}\left(\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2}\right) = \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2+2\cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right) = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \qquad \left(\because 1+\cos x = 2\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) == \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = RHS$$

20. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

Ans:

Let $x = \cos y \Rightarrow y = \cos^{-1}x$

$$LHS = \tan^{-1}\left(\frac{\sqrt{1+\cos y}-\sqrt{1-\cos y}}{\sqrt{1+\cos y}+\sqrt{1-\cos y}}\right) = \tan^{-1}\left(\frac{2\cos\frac{y}{2}-2\sin\frac{y}{2}}{2\cos\frac{y}{2}+2\sin\frac{y}{2}}\right)$$

$$\left(\because 1+\cos y = 2\cos^2\frac{y}{2} \;\; and \;\; 1-\cos y = 2\sin^2\frac{y}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{y}{2}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\left(\because \tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}\right)$$

21. Solve for x: $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

Ans:

Given $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

$$\Rightarrow 2\tan^{-1}\frac{1-x}{1+x} = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}x \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{1+2x+x^2-1+2x-x^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{4x}\right) = \tan^{-1}x \Rightarrow \tan^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}x$$

$$\Rightarrow \frac{1-x^2}{2x} = x \Rightarrow 1-x^2 = 2x^2 \Rightarrow 1 = 3x^2 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$\left[\because x>0 \text{ given, so we do not take } x = -\frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

22. Solve for x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Ans:

Given $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \Rightarrow \frac{\cos x}{\sin x} = 1$$

$$\Rightarrow \cot x = 1 \Rightarrow \cot x = \cot\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$$

23. Solve for x: $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

Ans:

Given $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x) \Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$$

$$\left[\because \sin^{-1}(1-x) + \cos^{-1}(1-x) = \frac{\pi}{2}\right]$$

$$\Rightarrow \cos\left(-2\sin^{-1}x\right) = 1-x$$

$$\Rightarrow \cos\left(2\sin^{-1}x\right) = 1-x \qquad [\because \cos(-x) = \cos x]$$

$$\Rightarrow 1 - 2\sin^2\left(\sin^{-1}x\right) = 1-x \qquad \left[\because \cos 2x = 1 - 2\sin^2 x\right]$$

$$\Rightarrow 1 - 2\left[\sin\left(\sin^{-1}x\right)\right]^2 = 1-x$$

$$\Rightarrow 1 - 2x^2 = 1 - x \Rightarrow 2x^2 - x = 0$$

$$\Rightarrow x(2x-1) = 0 \Rightarrow x = 0 \;\; or \;\; 2x - 1 = 0$$

$$\Rightarrow x = 0 \;\; or \;\; x = \frac{1}{2}$$

But $x = \frac{1}{2}$ does not satisfy the given equation, so $x = 0$.

24. Simplify: $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

Ans:

Given $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left(\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}1\right) \qquad \left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}1 = \frac{\pi}{4}$$

25. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ **in the simplest form.**

Ans:

Given $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\cos\frac{x}{2}\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\right)$$

$$\left(\because 1-\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}, \sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \;\; and \;\; \sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{\pi}{4}+\frac{x}{2}$$

26. Simplify : $\cot^{-1}\frac{1}{\sqrt{x^2-1}}, |x|>1$

Ans:

Let $x = \sec\theta$, then $\theta = \sec^{-1}x$ ………………… (i)

$$\cot^{-1}\frac{1}{\sqrt{x^2-1}}=\cot^{-1}\frac{1}{\sqrt{\sec^2\theta-1}}=\cot^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$=\cot^{-1}\frac{1}{\tan\theta}=\cot^{-1}(\cot\theta)=\theta=\sec^{-1}x$$

27. Prove that $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$

Ans:

Let $\sin^{-1}\frac{3}{5}=x$ and $\sin^{-1}\frac{8}{17}=y$

Therefore $\sin x=\frac{3}{5}$ and $\sin y=\frac{8}{17}$

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$

and $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{1-\frac{64}{289}}=\frac{15}{17}$

We have $\cos(x-y)=\cos x\cos y+\sin x\sin y=\frac{4}{5}\times\frac{15}{17}+\frac{3}{5}\times\frac{8}{17}=\frac{60}{85}+\frac{24}{85}=\frac{84}{85}$

$\Rightarrow x-y=\cos^{-1}\frac{84}{85}$

$\Rightarrow \sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$

28. Prove that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Ans:

Let $\sin^{-1}\frac{12}{13}=x$, $\cos^{-1}\frac{4}{5}=y$ and $\tan^{-1}\frac{63}{16}=z$

Then $\sin x=\frac{12}{13}$, $\cos y=\frac{4}{5}$ and $\tan z=\frac{63}{16}$

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{1-\frac{144}{169}}=\frac{5}{13}$

and $\sin y=\sqrt{1-\cos^2 y}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

$$\Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5} \quad and \quad \tan y = \frac{\sin y}{\cos y} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x.\tan y} = \frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5}.\frac{3}{4}} = \frac{\frac{48+15}{20}}{1 - \frac{36}{20}} = \frac{\frac{63}{20}}{-\frac{16}{20}} = -\frac{63}{16} = -\tan z$$

$$\Rightarrow \tan(x+y) = -\tan z = \tan(-z) = \tan(\pi - z)$$

$$\Rightarrow x + y = \pi - z$$

$$\Rightarrow x + y + z = \pi$$

$$\Rightarrow \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$$

29. Simplify: $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), if \ \frac{a}{b}\tan x > -1$

Ans:

$$\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$$

30. Solve: $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

Ans:

Given $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right) = \frac{\pi}{4} \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow (6x-1)(x+1) = 0$$

$$\Rightarrow x = \frac{1}{6} \quad or \quad x = -1$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative,

$x = \frac{1}{6}$ is the only solution of the given equation.

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. Evaluate : $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Ans:

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\frac{\pi}{2}=1$$

2. Write the value of cot ($\tan^{-1}a + \cot^{-1}a$).

Ans:

$$\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}-\cot^{-1}a+\cot^{-1}a\right)\;=\cot\frac{\pi}{2}=0$$

3. Find the principal values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$**.**

Ans:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(\pi+\frac{\pi}{6}\right)\right)$$

$$=\cos^{-1}\left(-\cos\frac{\pi}{6}\right)=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$$

4. Find the principal values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Ans:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{4}\right)\right)$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\tan^{-1}(-1)=-\frac{\pi}{4}$$

5. Prove that: $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$

Ans:

$$LHS=\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}+\frac{\tan\frac{\pi}{4}-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1+\tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$=\frac{1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}+\frac{1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{\left[1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2+\left[1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2}{\left[1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]}$$

$$= \frac{2+2\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2\left(1+\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\cos\left(\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\frac{a}{b}} = \frac{2b}{a} = RHS$$

6. **Solve:** $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$

Ans:

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1).(x-1)}\right)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-(x^2-1)}\right)=\tan^{-1}\frac{8}{31} \Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \frac{2x}{2-x^2}=\frac{8}{31} \Rightarrow 62x=16-8x^2 \Rightarrow 8x^2+62x-16=0$$

$$\Rightarrow 4x^2+31x-8=0 \Rightarrow (4x-1)(x+8)=0$$

$$\Rightarrow x=\frac{1}{4} \;\; and \;\; x=-8$$

As $x=-8$ does not satisfy the equation

Hence $x=\frac{1}{4}$ is only solution.

7. **Prove that** $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

Ans:

$$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\frac{\pi}{2}-\sin^{-1}\frac{16}{65}=\cos^{-1}\frac{16}{65}$$

Let $\sin^{-1}\frac{4}{5}=x$ and $\sin^{-1}\frac{5}{13}=y$

Therefore $\sin x=\frac{4}{5}$ and $\sin y=\frac{5}{13}$

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

and $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{144}}=\frac{12}{13}$

We have $\cos(x+y)=\cos x\cos y-\sin x\sin y=\frac{3}{5}\times\frac{12}{13}-\frac{4}{5}\times\frac{5}{13}=\frac{36}{65}-\frac{20}{65}=\frac{16}{65}$

$$\Rightarrow x+y=\cos^{-1}\frac{16}{65}$$

$$\Rightarrow \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\cos^{-1}\frac{16}{65}$$

8. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$

Ans:

$$LHS=\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}$$

$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}.\frac{2}{9}}\right)=\tan^{-1}\left(\frac{\frac{9+8}{36}}{1-\frac{2}{36}}\right)=\tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right) \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\frac{17}{34}=\tan^{-1}\frac{1}{2}=\frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right)$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right)=\frac{1}{2}\cos^{-1}\frac{3}{5}=RHS$$

9. Solve for x: $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$

Ans:

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \cos^{-1}\left(\frac{-(1-x^2)}{1+x^2}\right)+\tan^{-1}\left(-\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \pi-2\tan^{-1}x-2\tan^{-1}x=\frac{2\pi}{3}\Rightarrow \pi-4\tan^{-1}x=\frac{2\pi}{3}$$

$$\Rightarrow \pi-\frac{2\pi}{3}=4\tan^{-1}x\Rightarrow 4\tan^{-1}x=\frac{\pi}{3}\Rightarrow \tan^{-1}x=\frac{\pi}{12}$$

$$\Rightarrow \tan^{-1}x=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}.\tan\frac{\pi}{6}}$$

$$\Rightarrow x=\frac{1-\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\Rightarrow x = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3+1-2\sqrt{3}}{3-1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

10. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0,1)$

Ans:

$$LHS = \tan^{-1}\sqrt{x} = \frac{1}{2}\left(2\tan^{-1}\sqrt{x}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right) \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = RHS$$

11. Prove that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Ans:

$$LHS = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \frac{9}{4}\cos^{-1}\frac{1}{3}$$

$$Let \ \cos^{-1}\frac{1}{3} = x \Rightarrow \cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\cos^2 x}$$

$$\Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow x = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\therefore \frac{9}{4}\cos^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = RHS$$

12. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

Ans:

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(-\sec\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \sec^{-1}\left(\sec\left(\pi - \frac{\pi}{3}\right)\right) = \frac{\pi}{3} - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

13. Prove that : $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Ans:

Let $\sin^{-1}\frac{3}{5} = x$ and $\cot^{-1}\frac{3}{2} = y$

Then $\sin x = \frac{3}{5}$ and $\cot y = \frac{3}{2}$

Now $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\sin y = \frac{1}{\sqrt{1+\cot^2 y}} = \frac{1}{\sqrt{1+\left(\frac{3}{2}\right)^2}} = \frac{1}{\sqrt{1+\frac{9}{4}}} = \frac{1}{\sqrt{\frac{13}{4}}} = \frac{2}{\sqrt{13}}$

$\Rightarrow \cos y = \frac{3}{\sqrt{13}}$

$LHS = \cos(x+y) = \cos x \cos y - \sin x \sin y = \frac{4}{5}\times\frac{3}{\sqrt{13}} - \frac{3}{5}\times\frac{2}{\sqrt{13}}$

$= \frac{12}{5\sqrt{13}} - \frac{6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}} = RHS$

Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

Ans:

Let $2\tan^{-1}\frac{1}{5} = x \Rightarrow \tan^{-1}\frac{1}{5} = \frac{x}{2} \Rightarrow \tan\frac{x}{2} = \frac{1}{5}$

$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{2}{5}\times\frac{25}{24} = \frac{5}{12}$

14. Find the value of the following: $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0 \;\; and \;\; xy<1$

Ans:

Let $x = \tan\alpha \;\; and \;\; y = \tan\beta \Rightarrow \alpha = \tan^{-1}x, \beta = \tan^{-1}y$

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= \tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha} + \cos^{-1}\frac{1-\tan^2\beta}{1+\tan^2\beta}\right]$

$= \tan\frac{1}{2}\left[\sin^{-1}(\sin 2\alpha) + \cos^{-1}(\cos 2\beta)\right] \quad \left[\because \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \;\; and \;\; \cos 2\beta = \frac{1-\tan^2\beta}{1+\tan^2\beta}\right]$

$= \tan\frac{1}{2}\left[2\alpha + 2\beta\right] = \tan\left[\alpha+\beta\right] = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha.\tan\beta} = \frac{x+y}{1-xy}$

15. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

Ans:

$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= \tan^{-1}\left[2\sin\left(2\frac{\pi}{6}\right)\right] = \tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right] = \tan^{-1}\left[2\times\frac{\sqrt{3}}{2}\right]$

$= \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

16. Prove that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

Ans:

$Let\ \ \sin^{-1}\frac{3}{4}=\alpha \Rightarrow \sin\alpha=\frac{3}{4}$

$\Rightarrow \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\frac{3}{4} \qquad \left[\because \sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}\right]$

$\Rightarrow 3+3\tan^2\frac{\alpha}{2}=8\tan\frac{\alpha}{2} \Rightarrow 3\tan^2\frac{\alpha}{2}-8\tan\frac{\alpha}{2}+3=0$

$\Rightarrow \tan\frac{\alpha}{2}=\frac{8\pm\sqrt{64-36}}{6}=\frac{8\pm\sqrt{28}}{6}$

$\Rightarrow \tan\frac{\alpha}{2}=\frac{8\pm2\sqrt{7}}{6} \Rightarrow \tan\frac{\alpha}{2}=\frac{4\pm\sqrt{7}}{3}$

$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

17. If $y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$**, then prove that** $\sin y=\tan^2\frac{x}{2}$

Ans:

$y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$

$\Rightarrow y=\frac{\pi}{2}-\tan^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)=\frac{\pi}{2}-2\tan^{-1}\left(\sqrt{\cos x}\right)$

$\Rightarrow y=\frac{\pi}{2}-\cos^{-1}\left(\frac{1-\cos x}{1+\cos x}\right) \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

$\Rightarrow y=\sin^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)$

$\Rightarrow \sin y=\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan^2\frac{x}{2}$

18. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$**, then find the value of x.**

Ans:

$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1 \ \Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1$

$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2} \Rightarrow \sin^{-1}\frac{1}{5}=\frac{\pi}{2}-\cos^{-1}x=\sin^{-1}x$

$\Rightarrow x=\frac{1}{5}$

19. Prove that $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Ans:

$LHS=2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}$

$$= 2\tan^{-1}\frac{1}{5} + 2\tan^{-1}\frac{1}{8} + \sec^{-1}\frac{5\sqrt{2}}{7} = 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}.\frac{1}{8}}\right) + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$$

$$= 2\tan^{-1}\left(\frac{\frac{8+5}{40}}{1-\frac{1}{40}}\right) + \tan^{-1}\sqrt{\frac{50}{49}-1}$$

$$= 2\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right) + \tan^{-1}\sqrt{\frac{1}{49}} = 2\tan^{-1}\left(\frac{13}{39}\right) + \tan^{-1}\frac{1}{7}$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right) = \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

20. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$ **then write the value of** $x + y + xy$**.**

Ans:

$$\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-x.y}\right) = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-x.y} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow x + y = 1 - xy \Rightarrow x + y + xy = 1$$