KVS Junior Mathematics Olympiad (JMO)

SAMPLE PAPER – 8

M.M. 100 Time : 3 hours

Note : Attempt all questions.
All questions carry equal marks

Q1. If $y = |x-1| + |x-2|$
Express it without using modulus sign

Q2. If $x + \frac{1}{x} = -1$, then find the value of $x^{99} + \frac{1}{x^{99}}$

Q3. Show that there cannot exist any positive integral pair (x,y) satisfying the equation $x^2 = y^2 + 2182$

Q4. A rectangular courtyard 20m 16cm long and 15m 50 cm broad. It is to be paved with square tiles of same size. Find the minimum number of such tiles.

Q5. Which is greater : 11^{33} or $(2001)^{10}$

Q6. Factorize :
$(a + 1)^4 + (a^2 – 1)^2 + (a-1)^4$

Q7. The rodents eat through a 3x3x3 cube of eatable by tunneling through all the 27, 1x1x1 sub-cubes. If rodents start at one of the corner small cubes and always moves onto an uneaten adjacent small-cube, can it finish at the center of the cube ?

Q8. The four digit number aabb is a square number. Find it.

Q9. Four pipes, each of a diameter 1 metre, are tied together in a “square” shape by a band of metal. How long is the band of metal ?

Q10. In a triangle ABC, CD is altitude. Find the relationship between angles A and B, it is known that $CD^2 = AD.DB$.

SOLUTION AND HINTS (SAMPLE PAPER 8)

Q1. $y = |x-1| + |x-2|$

Let, $x < 1$ then $x - 1$ and $x-2$ are negative

$\therefore y = -(x-1) - (x-2)$

$= 3 - 2x$

Let, $1 \le x \le 2$ so that $x - 1 \ge 0$ and $x - 2 \le 0$

$\therefore y = -(x-1) + (2-x)$

$= 1$

Let $x > 2$ then $x - 1 > 0$ and $x - 2 > 0$

$\therefore y = -(x-1) - (x-2)$

$= 2x - 3$

$$\therefore y = \begin{cases} 3 - 2x \text{ when } x < 1 \\ 1 \text{ when } 1 \le x \le 2 \\ 2x - 3 \text{ when } x > 2 \end{cases}$$

Q2 If $x + \frac{1}{x} = -1$ Let $a_n = x^n + \frac{1}{x^n}$

So, $a_{n+1} = x^{n+1} + \frac{1}{x^{n+1}}$

$= x^n (x + \frac{1}{x}) - x^{n-1} - \frac{1}{x^{n+1}}$

$= x^n a_1 + x^{-n} (x + \frac{1}{x}) - x^{-n+1}$

$= x^n a_1 + \frac{a_1}{x^n} - (x^{n-1} + \frac{1}{x^{n-1}})$

$= a_1 a_n - a_{n-1}$

$a_1 = -1$, $a^2 = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = -1;$

$a_3 = a_1 a_2 - a_1$

$= 1 + a$

$= 2$

$a_4 = a_1 a_3 - a_2$

$= -2 + 1$

$= -1$

$a_{99} = (x^{33})^3 + \left(\frac{1}{x^{33}}\right)^3$

$= \left(x^{33} + \frac{1}{x^{33}}\right)^3 - 3 \left(x^{33} + \frac{1}{x^{33}}\right)^3$ (1)

Here, $x^{33} + \frac{1}{x^{33}} = (x^{11})^3 + \left(\frac{1}{x^{11}}\right)^3$

$= \left(x^{11} + \frac{1}{x^{11}}\right)^3$ - 3 $\left(x^{11} + \frac{1}{x^{11}}\right)^3$ (2)

$a_9 = x^9 + \frac{1}{x^9}$

$= (x^3)^3 + \left(\frac{1}{x^3}\right)^3$

$= \left(x^3 + \frac{1}{x^3}\right)^3$ -3 $\left(x^3 + \frac{1}{x^3}\right)$

$= 2^3 - 6$

$= 2$

$a_{10} = a_1\ a_9 - a_8$

$= -2 \left(x^8 + \frac{1}{x^8}\right)$

$= -2- \left[\left(x^4 + \frac{1}{x^4}\right)^2\right]$

$= - 2 - [1-2]$

$= -1$

$\therefore\ a_{11} = a_1 a_{10} - a_9$

$= 1 - 2$

$= -1$

from (2) : $a_{33} = -1 - 3 \text{ x } (-1)$ $= 2$

from (1) : $a_{99} = 2^3 - 3 \text{ x } 2$ $= 2$

Q3. $x^2 = y^2 + 2182$

$\Rightarrow x^2 - y^2 = 2182$

$\Rightarrow (x+y)\ (x-y) = 2182$

i.e. (x+y) (x-y) is even

means both (x+y) and (x-y) are both even.

But 2182 is not divisible by 4.

So no positive integers satisfy the equation.

Q4. A rectangular courtyard is 20m 16 cm long and 15 m 60 cm board. It is to be paved with square tiles of same size. Find the minimum number of such tiles.

A. The area = 2016 x 1560 cm^2

$(2 \text{ x } 2 \text{ x } 2 \text{ x } 2)^2 \text{ x } (3)^2 \text{ x } 3 \text{ x } 5\ 7 \text{ x } 13$

$= (48)^2 \text{ x } 3 \text{ x } 5 \text{ x } 7 \text{ x } 13$

$= (48)^2 \text{ x } 1365$

so square tiles of size 48 cm will be required in 1365 number.

Q5. We know,

$11^{33} = (11^3)^{11} > (11 \text{x} 10^2)^{11} > (2^{10})^{11} = (2^{11})^{10} > (2001)^{10}$

So $11^{33} > (2001)^{10}$

Q6. $(a+1)^4 + (a^2 - 1)^2 + (a-1)^4$

$= (a+1)^4 + (a + 1)^2 \ (a-1)^2 + (a-1)^4$

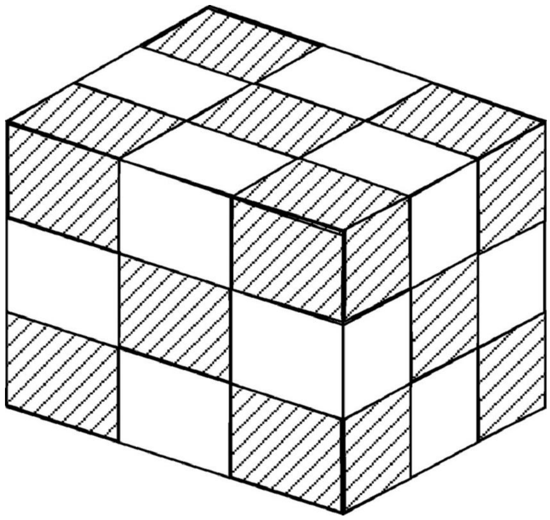
$= \{(a+1)^2 + (a + 1) \ (a-1) + (a-1)^2\}$

$\text{x} \ \{(a+1)^2 - (a + 1) \ (a-1) + (a-1)^2\}$

$= \{a^2 + 2a + 1 + a^2 - a + a^2 - 2a + 1\} \{a^2 + 2a + 1 - a^2 + 1 + a^2 - 2a + 1\}$

$= (3a^2 + 1) \ (a^2 + 3)$

Q7.



In the figure of 3 x 3 cube. The alternate small cubes are painted. The mouse will end up with alternate cube (small). If corner small cube remains unpainted in big cube there are 13 unpainted and 14 painted cubes.

The mouse must go

PUPU…..PUP (p – painted, u – unpainted)

So, last small cube should be painted.

So mouse cannot end up in center small cube.

Q8. If the number = N

$N = 1000x + 100x + 10y + y = 11 \ (100x + y)$

N is a perfect square divisible by 11.

So, $100x + y$ is divisible by 11.

$\Rightarrow 100x + y = 9 \text{ x } 11x + x + y$

so $x + y$ is divisible by 11.

x and $y \leq 9$

and x and $y \neq 0$

so, $1 \leq x + y \leq 18$

so $x + y = 11$

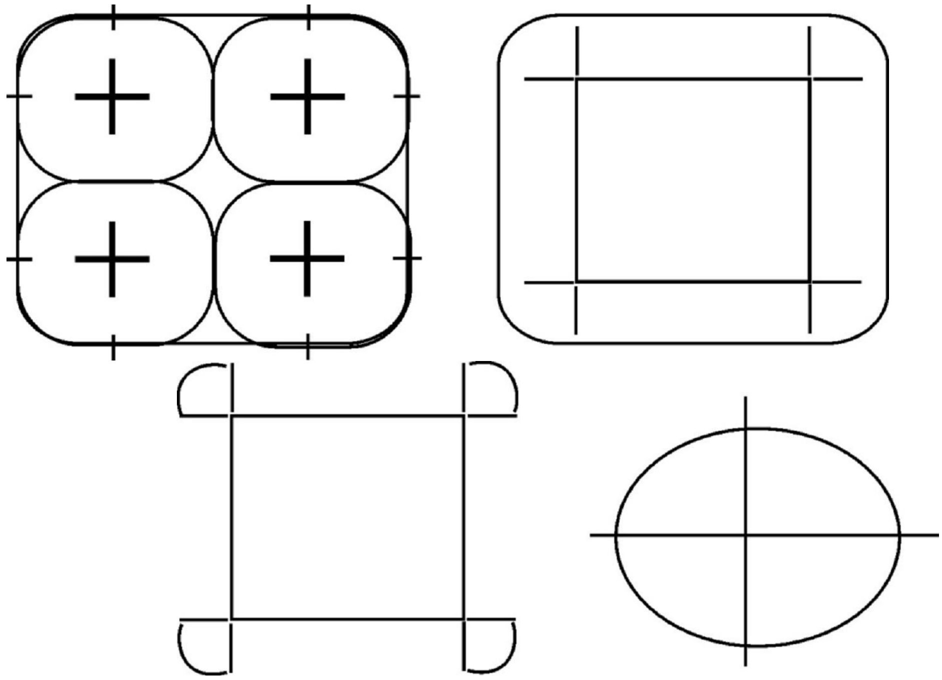
$\therefore N = 11 \ (100x + y) \quad = 11 \ \{99x + (x+y)\} = 11 \ (99x + 11) = (11)^2 \ (9x + 1)$

so $9x + y$ must be a perfect square and $1 \leq x \leq 9$

so $9x + y$ is perfect square when $x = 7$

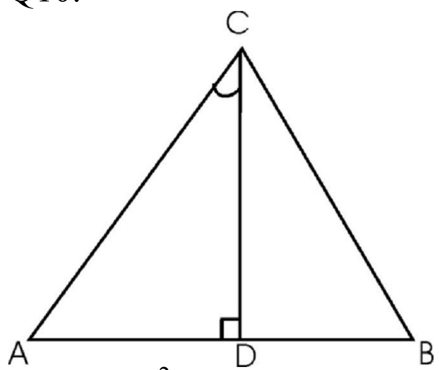
$\therefore N = (11)^2 \text{ x } 64 = 7744$

Q.9



So length of band = $2\pi.1 + 4$
= $(4+2\pi)$ meters

Q10.



Given $CD^2 = AD.DB$
$\Rightarrow \frac{CD}{DB} = \frac{AD}{CD}$
So, $\Delta ACD \sim \Delta CBD$
$\therefore \angle ACD = \angle CBD$ and $\angle CAD = \angle BCD$
Let $\angle ACD = \angle CBD = x$
and $\angle CAD = \angle BCD = y$

we have
$\angle ACD + \angle CBD + \angle CAD + \angle BCD = 180^o$
$\Rightarrow 2x + 2y = 180^o$
$x + y = 90^o$
$\therefore \angle A + \angle B = 90^o$