CHAPTER – 4: DETERMINANTS

MARKS WEIGHTAGE – 10 marks

NCERT Important Questions & Answers

1. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$**, then find the value of** ***x*****.**

Ans:

Given that $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

On expanding both determinants, we get

$x \times x - 18 \times 2 = 6 \times 6 - 18 \times 2 \Rightarrow x^2 - 36 = 36 - 36$

$\Rightarrow x^2 - 36 = 0 \Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

2. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

Ans:

Applying operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ to the given determinant Δ, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we obtain

$$\Delta = a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0 = a(a^2 - 0) = a(a^2) = a^3$$

3. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Ans:

$$Let\ \Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2 - R_3$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$= 2c(ab + b^2 - bc) - 2b(bc - c^2 - ac)$
$= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc$
$= 4abc$

4. If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ **then show that $1 + xyz = 0$**

Ans:

We have $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$

Now, we know that If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

$\therefore \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$

$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ (Using $C_3 \leftrightarrow C_2$ and then $C_1 \leftrightarrow C_2$)

$= (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$= (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$ (Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Taking out common factor $(y - x)$ from R_2 and $(z - x)$ from R_3, we get

$$\Delta = (1 + xyz)(y - x)(z - x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$= (1 + xyz)(y - x)(z - x)(z - y)$ (on expanding along C_1)

Since $\Delta = 0$ and x, y, z are all different, i.e., $x - y \neq 0$, $y - z \neq 0$, $z - x \neq 0$, we get
$1 + xyz = 0$

5. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = abc + bc + ca + ab$

Ans:

$LHS = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking out factors a,b,c common from R_1, R_2 and R_3, we get

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left[1(1-0)\right]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab = \text{RHS}$$

6. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Ans:

$$LHS = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} \quad \text{(interchange row and column)}$$

$$= \begin{vmatrix} b+c & c+a & -2c \\ q+r & r+p & -2r \\ y+z & z+x & -2z \end{vmatrix} \quad [\text{using} C_3 \to C_3 - (C_1 + C_2)]$$

$$= (-2)\begin{vmatrix} b+c & c+a & c \\ q+r & r+p & r \\ y+z & z+x & z \end{vmatrix} \quad \text{(taking '–2' common from } C_3\text{)}$$

$$=(-2)\begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix}$$ (using $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$)

$$=2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ (using $C_1 \leftrightarrow C_2$)

$$=2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = RHS$$ (interchange row and column)

7. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Ans:

$$LHS = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ [taking out factors a from R_1, b from R_2 and c from R_3]

$$=(abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ (taking out factors a from C_1, b from C_2 and c from C_3)

$$=a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$ (using $R_1 \to R_1 + R_2$ and $R_2 \to R_2 - R_3$)

Expanding corresponding to first row R_1, we get

$$=a^2b^2c^2 \times 2\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}$$

$$=a^2b^2c^2 \times 2(0+2) = 4a^2b^2c^2 = RHS$$

8. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Ans:

$$LHS = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$ *and* $R_2 \to R_2 - R_3$, we get

$$=\begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-c & (a-c)(a+c) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Taking common factors $(a - c)$ and $(b - c)$ from R_1 and R_2 respectively, we get

$$= (a-c)(b-c)\begin{vmatrix} 0 & 1 & (a+c) \\ 0 & 1 & (b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Now, expanding corresponding to C_1, we get

$= (a - c) (b - c) (b + c - a - c) = (a - b) (b - c) (c - a) =$ RHS

9. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Ans:

$$LHS = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ *and* $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

Taking common $(a - b)$ from C_1 and $(b - c)$ from C_2, we get

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a^2+ab+b^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Now, expanding along R_1, we get

$= (a - b) (b - c) [1 \times (b^2 + bc + c^2) - 1 \times (a^2 + ab + b^2)]$

$= (a - b) (b - c) [b^2 + bc + c^2 - a^2 - ab - b^2]$

$= (a - b) (b - c) (bc - ab + c^2 - a^2)$

$= (a - b) (b - c) [b(c - a) + (c - a) (c + a)]$

$= (a - b) (b - c) (c - a) (a + b + c)=$ RHS.

10. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Ans:

$$LHS = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 \to xR_1, R_2 \to yR_2$ *and* $R_3 \to zR_3$, we have

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$ (take out xyz common from C_3)

$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$ (using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding corresponding to C_3, we get

$= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$

$= \left[(y^2 - x^2)(z^3 - x^3) - (z^2 - x^2)(y^3 - x^3) \right]$

$= (y + x)\,(y - x)\,(z - x)\,(z^2 + x^2 + xz) - (z + x)\,(z - x)\,(y - x)\,(y^2 + x^2 + xy)$

$= (y - x)\,(z - x)\,[(y + x)\,(z^2 + x^2 + xz) - (z + x)\,(y^2 + x^2 + xy)]$

$= (y - x)(z - x)[yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y]$

$= (y - x)(z - x)[yz^2 - zy^2 + xz^2 - xy^2]$

$= (y - x)(z - x)[yz(z - y) + x(z^2 - y^2)]$

$= (y - x)(z - x)[yz(z - y) + x(z - y)(z + y)]$

$= (y - x)\,(z - x)\,[(z - y)\,(xy + yz + zx)]$

$= (x - y)\,(y - z)\,(z - x)\,(xy + yz + zx) =$ RHS.

11. Using the property of determinants and without expanding, prove that

$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

Ans:

$LHS = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$ (using $C_1 \to C_1 + C_2 + C_3$)

$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$ [take out $(5x + 4)$ common from C_1].

$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x+4 & 0 \\ 0 & 0 & -x+4 \end{vmatrix}$ (Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_1, we get

$= (5x + 4)\,\{1(4 - x)\,(4 - x)\}$

$= (5x+4)(4-x)^2 =$ RHS.

12. Using the property of determinants and without expanding, prove that

$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Ans:

$$LHS = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix} \quad \text{(using } C_1 \to C_1 + C_2 + C_3 \text{)}$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix} \quad \text{[take out } (5x + 4) \text{ common from } C_1\text{].}$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} \quad \text{(Using } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

Expanding along C_3, we get

$= (3y+k)\left[1(k^2 - 0)\right]$

$= k^2(3y+k)$ = RHS

13. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Ans:

$$LHS = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad \text{(Using } R_1 \to R_1 + R_2 + R_3\text{)}$$

Take out $(a + b + c)$ common from R_1, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & c-a-b \end{vmatrix} \quad \text{(Using } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\text{)}$$

Expanding along R_1, we get

$= (a + b + c)\ \{1(-b - c - a)\ (-c - a - b)\}$

$= (a + b + c)\ [-(b + c + a) \times (-)\ (c + a + b)]$

$= (a+b+c)(a+b+c)(a+b+c) = (a+b+c)^3$ = RHS

14. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Ans:

$$LHS = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$ (using $C_1 \to C_1 + C_2 + C_3$)

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$ [take out 2(*x* + y + z) common from C_1].

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$ (Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= 2(x+y+z)(x+y+z)(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we get

$$= 2(x+y+z)(x+y+z)(x+y+z)\left[1(1-0)\right]$$

$$= 2(x+y+z)(x+y+z)(x+y+z) = 2(x+y+z)^3 = \text{RHS}$$

15. Using the property of determinants and without expanding, prove that $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \left(1-x^3\right)^2$

Ans:

$$LHS = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$$ (using $C_1 \to C_1 + C_2 + C_3$)

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$ [take out $(1+x+x^2)$ common from C_1].

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$ (Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x(1-x) \\ 0 & x(x-1) & 1-x^2 \end{vmatrix}$$

Take out $(1-x)$ common from R_2 and same from R_3, we get

$$= (1+x+x^2)(1-x)(1-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

Expanding along C_1, we get

$$= (1+x+x^2)(1-x)(1-x)\begin{vmatrix} 1 & x \\ -x & 1+x \end{vmatrix}$$

$$= (1+x+x^2)(1-x)(1-x)(1+x+x^2)$$

$$= \left(1-x^3\right)\left(1-x^3\right) = \left(1-x^3\right)^2 = \text{RHS}$$

16. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Ans:

$$LHS = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} \qquad (\text{Using } C_1 \to C_1 - bC_3 \text{ and } C_2 \to C_2 + aC_3)$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix} \qquad (R_3 \to R_3 - bR_1 + aR_2)$$

Expanding along R_1, we get

$$= (1+a^2+b^2)^2\left[1(1+a^2+b^2)\right]$$

$$= (1+a^2+b^2)^3 = RHS$$

17. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Ans:

$$LHS = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors a, b and c from R_1, R_2 and R_3 respectively, we get

$$= \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

$$= \begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix} \qquad \text{(Using } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

Multiply and divide C_1 by a, C_2 by b and C_3 by c and then take common out from C_1, C_2 and C_3 respectively, we get

$$= abc \times \frac{1}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we get

$$= -1 \times (-c^2) + 1\left[1(a^2+1) + 1(b^2)\right]$$

$$= 1 + a^2 + b^2 + c^2 = RHS$$

18. Find values of k if area of triangle is 4 sq. units and vertices are (i) (k, 0), (4, 0), (0, 2) (ii) (–2, 0), (0, 4), (0, k)

Ans:

(i) We have Area of triangle = $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$

$\Rightarrow |\, k(0-2) + 1(8-0)| = 8$

$\Rightarrow k(0-2) + 1(8-0) = \pm 8$

On taking positive sign $-2k + 8 = 8$

$\Rightarrow -2k = 0$

$\Rightarrow k = 0$

On taking negative sign $-2k + 8 = -8$

$\Rightarrow -2k = -16$

$\Rightarrow k = 8$

$\Rightarrow k = 0, 8$

(ii) We have Area of triangle = $\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$

$\Rightarrow |-2(4-k) + 1(0-0)| = 8$

$\Rightarrow -2(4-k) + 1(0-0) = \pm 8$

$\Rightarrow [-8 + 2k] = \pm 8$

On taking positive sign, $2k - 8 = 8 \Rightarrow 2k = 16 \Rightarrow k = 8$

On taking negative sign, $2k - 8 = -8 \Rightarrow 2k = 0 \Rightarrow k = 0$

$\Rightarrow k = 0, 8$

19. If area of triangle is 35 sq units with vertices (2, – 6), (5, 4) and (*k*, 4). Then find the value of *k*.

Ans:

We have Area of triangle $=\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}=35$

$\Rightarrow |2(4-4)+6(5-k)+1(20-4k)|=70$

$\Rightarrow 2(4-4)+6(5-k)+1(20-4k)=\pm 70$

$\Rightarrow 30-6k+20-4k=\pm 70$

On taking positive sign, $-10k+50=70$

$\Rightarrow -10k=20 \Rightarrow k=-2$

On taking negative sign, $-10k+50=-70$

$\Rightarrow -10k=-120 \Rightarrow k=12$

$\therefore k=12, -2$

20. Using Cofactors of elements of second row, evaluate $\Delta=\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Ans:

Given that $\Delta=\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Cofactors of the elements of second row

$A_{21}=(-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix}=-(9-16)=7$

$A_{22}=(-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix}=(15-8)=7$

and $A_{23}=(-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}=-(10-3)=-7$

Now, expansion of Δ using cofactors of elements of second row is given by

$\Delta=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}$

$= 2 \times 7 + 0 \times 7 + 1(-7) = 14 - 7 = 7$

21. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$**, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.**

Ans:

Given that A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Now, $A^2 - 5A + 7I = O$

$A^2=A.A=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 9-2 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}-5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}-\begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}+\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$=\begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+3 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=O$

$\therefore$ $A^2 - 5A + 7I = O$

$\because |A| = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 6 + 1 = 7 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A.A - 5A = -7I$

Multiplying by A^{-1} on both sides, we get

$A.A\,(A^{-1}) - 5A(A^{-1}) = -7I(A^{-1})$

$\Rightarrow AI - 5I = -7A^{-1}$ (using $AA^{-1} = I$ and $IA^{-1} = A^{-1}$)

$$A^{-1} = -\frac{1}{7}(A - 5I) = \frac{1}{7}(5I - A) = \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

22. For the matrix A = $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, find the numbers *a* and *b* such that $A^2 + aA + bI = O$.

Ans:

Given that A = $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now, $A^2 + aA + bI = O$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If two matrices are equal, then their corresponding elements are equal.

$\Rightarrow$ $11 + 3a + b = 0$ …(i)

$8 + 2a = 0$ …(ii)

$4 + a = 0$ …(iii)

and $3 + a + b = 0$ …(iv)

Solving Eqs. (iii) and (iv), we get $4 + a = 0$

$\Rightarrow a = -4$

and $3 + a + b = 0$

$\Rightarrow 3 - 4 + b = 0 \Rightarrow b = 1$

Thus, $a = -4$ and $b = 1$

23. For the matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

Ans:

Given that A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$and \quad A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1(3+6) + 1(-1-4) = 3 - 9 - 5 = -11 \neq 0$$

$\therefore A^{-1}$ *exist*

$Now, A^3 - 6A^2 + 5A + 11I = O$

$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) + 11(IA^{-1}) = O$

$\Rightarrow AAI - 6AI + 5I + 11A^{-1} = O$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

24. Solve system of linear equations, using matrix method,

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{vmatrix} = 2(20+6) - 1(-10-0) + 1(6-0)$$

$= 52 + 10 + 6 = 68 \neq 0$

Thus, A is non-singular, Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$.

Cofactors of A are

$A_{11} = 20 + 6 = 26$,

$A_{12} = -(-10 + 0) = 10$,

$A_{13} = 6 + 0 = 6$

$A_{21} = -(-5 - 3) = 8$,

$A_{22} = -10 - 0 = -10$,

$A_{23} = -(6 - 0) = -6$

$A_{31} = (-2 + 4) = 2$,

$A_{32} = -(-4 - 2) = 6$,

$A_{33} = -8 - 2 = -10$

$$adj(A) = \begin{bmatrix} 26 & 10 & 6 \\ 8 & -10 & -6 \\ 2 & 6 & -10 \end{bmatrix}^T = \begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{68}\begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 26+24+18 \\ 10-30+54 \\ 6-18-90 \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 68 \\ 34 \\ -102 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-3}{2} \end{bmatrix}$$

$$Hence,\ x = 1,\ y = \frac{1}{2}\ and\ z = \frac{-3}{2}$$

25. Solve system of linear equations, using matrix method,

***x* – *y* + *z* = 4**

2*x* + *y* – 3*z* = 0

***x* + *y* + *z* = 2**

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\ and\ B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 + 3) - (-1)(2 + 3) + 1(2 - 1) = 4 + 5 + 1 = 10 \neq 0$$

Thus, A is non-singular, Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$.

Cofactors of A are

$A_{11} = 1 + 3 = 4$,

$A_{12} = -(2 + 3) = -5$,

$A_{13} = 2 - 1 = 1$,

$A_{21} = -(-1 - 1) = 2$,

$A_{22} = 1 - 1 = 0$,

$A_{23} = -(1 + 1) = -2$,

$A_{31} = 3 - 1 = 2$,

$A_{32} = -(-3 - 2) = 5$,

$A_{33} = 1 + 2 = 3$

$$adj(A) = \begin{bmatrix} 4 & 5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

26. Solve system of linear equations, using matrix method,

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$$

$= 10 + 15 + 15 = 40 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists. Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 4 + 1 = 5$,

$A_{12} = -(-2 - 3) = 5$,

$A_{13} = (-1 + 6) = 5$,

$A_{21} = -(-6 + 3) = 3$,

$A_{22} = (-4 - 9) = -13$,

$A_{23} = -(-2 - 9) = 11$,

$A_{31} = 3 + 6 = 9$,

$A_{32} = -(2 - 3) = 1$,

$A_{33} = -4 - 3 = -7$

$$adj(A) = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = -1$.

27. Solve system of linear equations, using matrix method,

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Ans:

The given system can be written as $AX = B$, where

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 4 & -5\\ 2 & -1 & 3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}7\\ -5\\ 12\end{bmatrix}$$

$$Here, |A| = \begin{vmatrix}1 & -1 & 2\\ 3 & 4 & -5\\ 2 & -1 & 3\end{vmatrix} = 1(12-5)-(-1)(9+10)+2(-3-8)$$

$= 7 + 19 - 22 = 4 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 12 - 5 = 7$,

$A_{12} = -(9 + 10) = -19$,

$A_{13} = -3 - 8 = -11$,

$A_{21} = -(-3 + 2) = 1$,

$A_{22} = 3 - 4 = -1$,

$A_{23} = -(-1 + 2) = -1$,

$A_{31} = 5 - 8 = -3$,

$A_{32} = -(-5 - 6) = 11$,

$A_{33} = 4 + 3 = 7$

$$adj(A)=\begin{bmatrix}7 & -19 & -11\\ 1 & -1 & -1\\ -3 & 11 & 7\end{bmatrix}^T = \begin{bmatrix}7 & 1 & -3\\ -19 & -1 & 11\\ -11 & -1 & 7\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{4}\begin{bmatrix}7 & 1 & -3\\ -19 & -1 & 11\\ -11 & -1 & 7\end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}7 & 1 & -3\\ -19 & -1 & 11\\ -11 & -1 & 7\end{bmatrix}\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}49-5-36\\ -133+5+132\\ -77+5+84\end{bmatrix}=\frac{1}{4}\begin{bmatrix}8\\ 4\\ 12\end{bmatrix}=\begin{bmatrix}2\\ 1\\ 3\end{bmatrix}$$

Hence, $x = 2$, $y = 1$ and $z = 3$.

28. If A = $\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$ **find A^{-1}. Using A^{-1}, Solve system of linear equations:**

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Ans:

The given system can be written as $AX = B$, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2\,(-4+4) - (-3)\,(-6+4) + 5\,(3-2)$$

$= 0 - 6 + 5 = -1 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = -4 + 4 = 0$,

$A_{12} = -(-6 + 4) = 2$,

$A_{13} = 3 - 2 = 1$,

$A_{21} = -(6 - 5) = -1$,

$A_{22} = -4 - 5 = -9$,

$A_{23} = -(2 + 3) = -5$,

$A_{31} = (12 - 10) = 2$,

$A_{32} = -(-8 - 15) = 23$,

$A_{33} = 4 + 9 = 13$

$$adj(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = 3$.

29. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Ans:

Let the prices (per kg) of onion, wheat and rice be Rs. x, Rs. y and Rs. z, respectively then

$4x + 3y + 2z = 60$, $2x + 4y + 6z = 90$, $6x + 2y + 3z = 70$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 4(12-12) - 3(6-36) + 2(4-24)$$

$= 0 + 90 - 40 = 50 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists. Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are,

$A_{11} = 12 - 12 = 0,$
$A_{12} = -(6 - 36) = 30,$
$A_{13} = 4 - 24 = -20,$
$A_{21} = -(9 - 4) = -5,$
$A_{22} = 12 - 12 = 0,$
$A_{23} = -(8 - 18) = 10,$
$A_{31} = (18 - 8) = 10,$
$A_{32} = -(24 - 4) = -20,$
$A_{33} = 16 - 6 = 10$

$$adj(A) = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 - 450 + 700 \\ 1800 + 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore$ $x = 5$, $y = 8$ and $z = 8$.

Hence, price of onion per kg is Rs. 5, price of wheat per kg is Rs. 8 and that of rice per kg is Rs. 8.

30. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Ans:

$$LHS = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

Applying $R_1 \to aR_1, R_2 \to bR_2$ *and* $R_3 \to cR_3$, we get

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$ [Taking out factor *abc* from C_3]

$$= (-1)^2\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$ (using $C_1 \leftrightarrow C_3$ and $C_2 \leftrightarrow C_3$)

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = RHS$$

31. If *a, b* and *c* are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$**. Show that either *a + b + c* = 0 or *a = b = c*.**

Ans:

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} \quad \text{(using } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} \quad \text{[take out } 2(a+b+c) \text{ common from } C_1\text{].}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} \quad \text{(Using } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

Expanding along C_1, we get

$$= 2(a+b+c)\begin{vmatrix} b-c & c-a \\ b-a & c-b \end{vmatrix}$$

$$= 2(a+b+c)\left[(b-c)(c-b)-(c-a)(b-a)\right]$$

$$= 2(a+b+c)\left[bc-b^2-c^2+bc-(bc-ac-ab+a^2)\right]$$

$$= 2(a+b+c)\left[bc-b^2-c^2+bc-bc+ac+ab-a^2\right]$$

$$= 2(a+b+c)\left[ab+bc+ac-a^2-b^2-c^2\right]$$

It is given that Δ= 0,

$$2(a+b+c)\left[ab+bc+ac-a^2-b^2-c^2\right]=0$$

$$\Rightarrow Either\ \ a+b+c=0\ \ or\ \ ab+bc+ac-a^2-b^2-c^2=0$$

$$\Rightarrow ab+bc+ac-a^2-b^2-c^2=0$$

$$\Rightarrow 2ab+2bc+2ac-2a^2-2b^2-2c^2=0$$

$$\Rightarrow 2a^2+2b^2+2c^2-2ab-2bc-2ac=0$$

$$\Rightarrow a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ac=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \quad \text{[since square of any real number is never negative]}$$

$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$$\Rightarrow a=b, b=c, c=a$$

$$\Rightarrow a=b=c$$

32. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Ans:

$$LHS = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out a from C_1, b from C_2 and c from C_3, we get

$$= abc\begin{vmatrix} a & b & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$ [Using $C_1 \to C_1 + C_2 - C_3$]

$$= abc\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix}$$ [Using $R_2 \to R_2 - R_3$]

Expanding along C_1,we get
$= (abc)\ [\ (2\ b)\ \{\ c(a-c) + c\ (a+c)\ \}\ ]$
$= 2\ (ab^2c)\ (2ac) = 4\ a^2b^2c^2$ = RHS.

33. Using properties of determinants, prove that $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Ans:

$$LHS = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha+\beta \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$ (using $C_3 \to C_3 + C_1$)

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$ (Taking out $(\alpha + \beta + \gamma)$ common from C_1)

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$$ (Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_3, we get
$= (\alpha + \beta + \gamma)\ [(\beta - \alpha)(\gamma^2 - \alpha^2) - (\gamma - \alpha)(\beta^2 - \alpha^2)]$
$= (\alpha + \beta + \gamma)\ [(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha) - (\gamma - \alpha)(\beta - \alpha)(\beta + \alpha)]$
$= (\alpha + \beta + \gamma)\ (\beta - \alpha)(\gamma - \alpha)[\gamma + \alpha - \beta - \alpha]$
$= (\alpha + \beta + \gamma)\ (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$
$= (\alpha + \beta + \gamma)\ (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$ = RHS

34. Using properties of determinants, prove that $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Ans:

$$LHS = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix} \quad \text{(using } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \quad \text{(Taking out } (a + b + c) \text{ common from } C_1\text{)}$$

Now applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1, we get

$= (a + b + c)[(2b + a)(2c + a) - (a - b)(a - c)]$

$= (a + b + c)[4bc + 2ab + 2ac + a - a + ac + ba - bc]$ 2 2

$= (a + b + c)(3ab + 3bc + 3ac) = 3(a + b + c)(ab + bc + ca) = RHS$

35. Solve the system of equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Ans:

Let $\frac{1}{x} = p,\ \frac{1}{y} = q \text{ and } \frac{1}{z} = r$, then the given equations become

$2p + 3q + 10r = 4$, $4p - 6q + 5r = 1$, $6p + 9q - 20r = 2$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36)$$

$= 150 + 330 + 720 = 1200 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the above system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 120 - 45 = 75$,

$A_{12} = -(-80 - 30) = 110,$
$A_{13} = (36 + 36) = 72,$
$A_{21} = -(-60 - 90) = 150,$
$A_{22} = (-40 - 60) = -100,$
$A_{23} = -(18 - 18) = 0,$
$A_{31} = 15 + 60 = 75,$
$A_{32} = -(10 - 40) = 30,$
$A_{33} = -12 - 12 = -24$

$$adj(A) = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow p = \frac{1}{2}, q = \frac{1}{3}, r = \frac{1}{5}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$$

$\Rightarrow x = 2$, $y = 3$ and $z = 5$.

36. If *a, b, c*, are in A.P, then find the determinant of $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

Ans:

$$Let\ \ A = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2(2b-a-c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad \text{(using } R_2 \rightarrow 2R_2 - R_1 - R_3\text{)}$$

But *a*,*b*, *c* are in AP. Using $2b = a + c$, we get

$$A = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix} = 0 \quad \text{[Since, all elements of } R_2 \text{ are zero]}$$

37. Show that the matrix $A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

Ans:

Given that $A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$

$$A^2 = AA=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}$$

$$Hence,\ A^2-4A+I=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}-4\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}-\begin{bmatrix}8 & 12\\ 4 & 8\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7-8+1 & 12-12+0\\ 4-4+0 & 7-8+1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

$Now,\ A^2-4A+I=O$

$\Rightarrow AA-4A=-I$

$\Rightarrow AA(A^{-1})-4AA^{-1}=-IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

$\Rightarrow A(AA^{-1})-4I=-A^{-1}$

$\Rightarrow AI-4I=-A^{-1}$

$$\Rightarrow A^{-1}=4I-A=4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}4-2 & 0-3\\ 0-1 & 4-2\end{bmatrix}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$$

38. Solve the following system of equations by matrix method.

$3x - 2y + 3z = 8$

$2x + y - z = 1$

$4x - 3y + 2z = 4$

Ans:

The system of equation can be written as $AX = B$, where

$$A=\begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ and\ B=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{vmatrix}$$

$= 3 (2 - 3) + 2(4 + 4) + 3 (- 6 - 4) = - 17 \neq 0$

Hence, A is nonsingular and so its inverse exists. Now

$A_{11} = -1$, $A_{12} = -8$, $A_{13} = -10$

$A_{21} = -5$, $A_{22} = -6$, $A_{23} = 1$

$A_{31} = -1$, $A_{32} = 9$, $A_{33} = 7$

$$adj(A)=\begin{bmatrix}-1 & -8 & 10\\ -5 & -6 & 1\\ -1 & 9 & 7\end{bmatrix}^T=\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x = 1$, $y = 2$ and $z = 3$.

39. Show that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

Ans:

Given that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix}$

Applying $R_1 \to xR_1$, $R_2 \to yR_2$, $R_3 \to zR_3$ to Δ and dividing by xyz, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common factors x, y, z from C_1, C_2 and C_3 respectively, we get

$$\Delta = \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have

$$\Delta = \begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+z)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix}$$

Taking common factor $(x + y + z)$ from C_2 and C_3, we have

$$\Delta = (x+y+z)^2\begin{vmatrix} (y+z)^2 & x-(y+z) & x-(y+z) \\ y^2 & (x+z)-y & 0 \\ z^2 & 0 & (x+y)-z \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, we have

$$\Delta = (x+y+z)^2\begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x-y+z & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

Applying $C_2 \to (C_2 + \frac{1}{y}C_1)$ and $C_3 \to C_3 + \frac{1}{z}C_1$, we get

$$\Delta = (x+y+z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+z & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x+y \end{vmatrix}$$

Finally expanding along R1, we have

$\Delta = (x + y + z)^2 (2yz) [(x + z) (x + y) – yz] = (x + y + z)^2 (2yz) (x^2 + xy + xz)$

$= (x + y + z)^3 (2xyz)$

40. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **to solve the system of equations**

$x – y + 2z = 1$

$2y – 3z = 1$

$3x – 2y + 4z = 2$

Ans:

Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Hence, \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$