CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. **Find the principal values of** $\sin^{-1}\left(-\frac{1}{2}\right)$**.**

Ans:

Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta \Rightarrow \sin\theta=-\frac{1}{2}$

We know that the range of principal value of $\sin^{-1}\theta$ *is* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin\theta=-\frac{1}{2}=\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right) \qquad \left(\because \sin(-\theta)=-\sin\theta\right)$

$\Rightarrow \theta=-\frac{\pi}{6}, where\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\Rightarrow \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

Hence the principal value of $\sin^{-1}\left(-\frac{1}{2}\right) is -\frac{\pi}{6}$

2. **Find the principal values of** $\tan^{-1}\left(-\sqrt{3}\right)$

Ans:

Let $\tan^{-1}\left(-\sqrt{3}\right)=\theta \Rightarrow \tan\theta=-\sqrt{3}$

We know that the range of principal value of $\tan^{-1}\theta$ *is* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \tan\theta=-\sqrt{3}=-\tan\frac{\pi}{3}=\tan\left(-\frac{\pi}{3}\right) \qquad \left(\because \tan(-\theta)=-\tan\theta\right)$

$\Rightarrow \theta=-\frac{\pi}{3}, where\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\Rightarrow \tan^{-1}\left(-\sqrt{3}\right)=-\frac{\pi}{3}$

Hence the principal value of $\tan^{-1}\left(-\sqrt{3}\right) is -\frac{\pi}{3}$

3. **Find the principal values of** $\cos^{-1}\left(-\frac{1}{2}\right)$

Ans:

Let $\cos^{-1}\left(-\frac{1}{2}\right)=\theta \Rightarrow \cos\theta=-\frac{1}{2}$

We know that the range of principal value of $\cos^{-1}\theta$ *is* $\left[0,\pi\right]$

$\therefore \cos\theta=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)$

$=\cos\frac{2\pi}{3} \qquad \left(\because \cos(\pi-\theta)=-\cos\theta\right)$

$\Rightarrow \theta=\frac{2\pi}{3}, where\ \ \theta\in\left[0,\pi\right]\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Hence the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ *is* $\frac{2\pi}{3}$

4. Find the principal values of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Ans:

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\theta \Rightarrow \cos\theta=-\frac{1}{\sqrt{2}}$

We know that the range of principal value of $\cos^{-1}\theta$ *is* $[0,\pi]$

$\therefore \cos\theta=-\frac{1}{\sqrt{2}}=-\cos\frac{\pi}{4}=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\frac{3\pi}{4} \qquad (\because \cos(\pi-\theta)=-\cos\theta)$

$\Rightarrow \theta=\frac{3\pi}{4}, where\ \ \theta\in[0,\pi] \Rightarrow \cos^{-1}\left(-\frac{1}{2}\right)=\frac{3\pi}{4}$

Hence the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ *is* $\frac{3\pi}{4}$

5. Find the principal values of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Ans:

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=\theta \Rightarrow \operatorname{cosec}\theta=-\sqrt{2}$

We know that the range of principal value of $\operatorname{cosec}^{-1}\theta$ *is* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

$\therefore \operatorname{cosec}\theta=-\sqrt{2}=-\operatorname{cosec}\frac{\pi}{4}=\operatorname{cosec}\left(-\frac{\pi}{4}\right) \qquad (\because \operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta)$

$\Rightarrow \theta=-\frac{\pi}{4}, where\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\} \Rightarrow \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=-\frac{\pi}{4}$

Hence the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) is -\frac{\pi}{4}$

6. Find the values of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Ans:

Let $\tan^{-1}(1)=x \Rightarrow \tan x=1=\tan\frac{\pi}{4} \Rightarrow x=\frac{\pi}{4} where\ \ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y \Rightarrow \cos y=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3} \qquad (\because \cos(\pi-\theta)=-\cos\theta)$

$\Rightarrow y=\frac{2\pi}{3} where\ \ y\in[0,\pi]$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z \Rightarrow \sin z=-\frac{1}{2}=-\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right) \Rightarrow z=-\frac{\pi}{6} where\ \ z\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)=x+y+z=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$$

7. Prove that $3\sin^{-1}x = \sin^{-1}(3x-4x^3), x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

Ans:

$Let\ \ \sin^{-1}x = \theta \Rightarrow x = \sin\theta,\ \ then$

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore 3\theta = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(3x-4x^3)$

$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

8. Prove that $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Ans:

$Given\ \ \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$LHS = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{48+77}{264}}{1-\frac{14}{264}}\right) = \tan^{-1}\left(\frac{\frac{125}{264}}{\frac{264-14}{264}}\right) = \tan^{-1}\left(\frac{\frac{125}{264}}{\frac{250}{264}}\right) = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = RHS$$

9. Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Ans:

$Given\ \ 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$LHS = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right) + \tan^{-1}\frac{1}{7} \qquad \left(\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

$$= \tan^{-1}\frac{1}{1-\frac{1}{4}} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{1-\frac{4}{21}}\right) = \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right) = \tan^{-1}\frac{31}{17} = RHS$$

10. Simplify : $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Ans:

Let $x = \tan\theta$, then $\theta = \tan^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{2} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \tan^{-1}\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\sec\theta - 1}{\tan\theta} = \tan^{-1}\left(\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \qquad \left[\begin{array}{l}\because 1-\cos\theta = 2\sin^2\frac{\theta}{2} \\ and \ \ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{array}\right]$$

$$= \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \quad \text{[using (i)]}$$

11. Simplify : $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Ans:

Let $x = \sec\theta$, then $\theta = \sec^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}} = \tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \tan^{-1}\frac{1}{\tan\theta} = \tan^{-1}(\cot\theta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)\left(\because \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta\right)$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1}x \qquad \text{[using (i)]}$$

12. Simplify : $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Ans:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right)$$

(inside the bracket divide numerator and denominator by cos x)

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) \qquad \left(\because \tan\left(\frac{\pi}{4} - x\right) = \frac{1-\tan x}{1+\tan x}\right)$$

$$= \frac{\pi}{4} - x$$

13. Simplify : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \ \ and \ \ xy < 1$

Ans:

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \ \ and \ \ xy < 1$$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} \ \ and \ \ 2\tan^{-1}y = \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \tan\frac{1}{2}\left[(2\tan^{-1}x + 2\tan^{-1}y)\right] = \tan\left[\frac{1}{2}.2(\tan^{-1}x + \tan^{-1}y)\right] = \tan(\tan^{-1}x + \tan^{-1}y)$$

$$= \tan\left(\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right) \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$= \dfrac{x+y}{1-x.y}$

14. If $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$**, find the value of x.**

Ans:

Given that $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\frac{\pi}{4} \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{1-\left(\frac{x^2-1}{x^2-4}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{\frac{(x^2+2x-x-2)+(x^2-2x+x-2)}{x^2-4}}{\left(\frac{x^2-4-x^2+1}{x^2-4}\right)}\right)=\tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{2x^2-4}{-3}\right)=1 \Rightarrow\ 2x^2-4=-3 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

15. Find the value of $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$**.**

Ans:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi-\frac{5\pi}{6}\right)\right) where, \frac{5\pi}{6}\in[0,\pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)=\frac{5\pi}{6}\ \ (\because \cos(2\pi-\theta)=\cos\theta)$$

16. Prove that $\cos^{-1}\dfrac{12}{13}+\sin^{-1}\dfrac{3}{5}=\sin^{-1}\dfrac{56}{65}$

Ans:

Given $\cos^{-1}\dfrac{12}{13}+\sin^{-1}\dfrac{3}{5}=\sin^{-1}\dfrac{56}{65}$

Let $\cos^{-1}\dfrac{12}{13}=x \Rightarrow \cos x=\dfrac{12}{13}$

$$\therefore \sin x=\sqrt{1-\cos^2 x}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$

$$\Rightarrow x=\sin^{-1}\frac{5}{13}$$

$$LHS=\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right) \quad \left[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right]$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{16}{25}}+\frac{3}{5}\sqrt{\frac{144}{169}}\right)=\sin^{-1}\left(\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right)$$

$$= \sin^{-1}\left(\frac{20}{65}+\frac{36}{65}\right)=\sin^{-1}\frac{56}{65}=RHS$$

17. Prove that $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Ans:

$$RHS=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$$

$$Let\ \sin^{-1}\frac{5}{13}=x \Rightarrow \sin x=\frac{5}{13}$$

$$\therefore \cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\Rightarrow \tan x=\frac{\sin x}{\cos x}=\frac{\frac{5}{13}}{\frac{12}{13}}=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$$

$$Let\ \cos^{-1}\frac{3}{5}=y \Rightarrow \cos y=\frac{3}{5}$$

$$\therefore \sin y=\sqrt{1-\cos^2 y}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\Rightarrow \tan y=\frac{\sin x}{\cos x}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$$

then the equation becomes $\tan^{-1}\frac{63}{16}=x+y$

$$\Rightarrow \tan^{-1}\frac{63}{16}=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}$$

$$RHS=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}}\right) \quad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\left(\frac{\frac{15+48}{36}}{1-\frac{20}{36}}\right)=\tan^{-1}\left(\frac{\frac{63}{36}}{\frac{16}{36}}\right)=\tan^{-1}\left(\frac{63}{16}\right)=LHS$$

18. Prove that $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Ans:

$$LHS=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}.\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}.\frac{1}{8}}\right) \qquad \left(\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{7+5}{35}}{1-\frac{1}{35}}\right)+\tan^{-1}\left(\frac{\frac{8+3}{24}}{1-\frac{1}{24}}\right) = \tan^{-1}\left(\frac{\frac{12}{35}}{\frac{30}{35}}\right)+\tan^{-1}\left(\frac{\frac{11}{24}}{\frac{23}{24}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}.\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{391}}{1-\frac{66}{391}}\right) = \tan^{-1}\left(\frac{\frac{325}{391}}{\frac{325}{391}}\right) = \tan^{-1}(1) = \frac{\pi}{4} = RHS$$

19. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

Ans:

Given $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

$$LHS = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right) \text{ (by rationalizing the denominator)}$$

$$= \cot^{-1}\left(\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2}\right) = \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2+2\cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right) = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \qquad \left(\because 1+\cos x = 2\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) == \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = RHS$$

20. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

Ans:

Let x = cosy $\Rightarrow y = \cos^{-1}x$

$$LHS = \tan^{-1}\left(\frac{\sqrt{1+\cos y}-\sqrt{1-\cos y}}{\sqrt{1+\cos y}+\sqrt{1-\cos y}}\right) = \tan^{-1}\left(\frac{2\cos\frac{y}{2}-2\sin\frac{y}{2}}{2\cos\frac{y}{2}+2\sin\frac{y}{2}}\right)$$

$$\left(\because 1+\cos y = 2\cos^2\frac{y}{2} \;\; and \;\; 1-\cos y = 2\sin^2\frac{y}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{y}{2}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\left(\because \tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}\right)$$

21. Solve for x: $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

Ans:

Given $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

$$\Rightarrow 2\tan^{-1}\frac{1-x}{1+x} = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}x \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{1+2x+x^2-1+2x-x^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{4x}\right) = \tan^{-1}x \Rightarrow \tan^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}x$$

$$\Rightarrow \frac{1-x^2}{2x} = x \Rightarrow 1-x^2 = 2x^2 \Rightarrow 1 = 3x^2 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$\left[\because x>0 \text{ given, so we do not take } x = -\frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

22. Solve for x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\cos ecx)$

Ans:

Given $2\tan^{-1}(\cos x) = \tan^{-1}(2\cos ecx)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\cos ecx) \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \Rightarrow \frac{\cos x}{\sin x} = 1$$

$$\Rightarrow \cot x = 1 \Rightarrow \cot x = \cot\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$$

23. Solve for x: $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

Ans:

Given $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x) \Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$$

$$\left[\because \sin^{-1}(1-x) + \cos^{-1}(1-x) = \frac{\pi}{2}\right]$$

$$\Rightarrow \cos\left(-2\sin^{-1}x\right) = 1-x$$

$$\Rightarrow \cos\left(2\sin^{-1}x\right) = 1-x \qquad [\because \cos(-x) = \cos x]$$

$$\Rightarrow 1 - 2\sin^2\left(\sin^{-1}x\right) = 1-x \qquad [\because \cos 2x = 1 - 2\sin^2 x]$$

$$\Rightarrow 1 - 2\left[\sin\left(\sin^{-1}x\right)\right]^2 = 1-x$$

$$\Rightarrow 1 - 2x^2 = 1 - x \Rightarrow 2x^2 - x = 0$$

$$\Rightarrow x(2x-1) = 0 \Rightarrow x = 0 \;\; or \;\; 2x - 1 = 0$$

$$\Rightarrow x = 0 \;\; or \;\; x = \frac{1}{2}$$

But $x = \frac{1}{2}$ does not satisfy the given equation, so $x = 0$.

24. Simplify: $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

Ans:

Given $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left(\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}1\right) \qquad \left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}1 = \frac{\pi}{4}$$

25. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ **in the simplest form.**

Ans:

Given $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\cos\frac{x}{2}\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\right)$$

$$\left(\because 1-\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2}, \sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \;\; and \;\; \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{\pi}{4}+\frac{x}{2}$$

26. Simplify : $\cot^{-1}\frac{1}{\sqrt{x^2-1}}, |x|>1$

Ans:

Let $x = \sec\theta$, then $\theta = \sec^{-1}x$ ………………… (i)

$$\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}\frac{1}{\sqrt{\sec^2\theta-1}} = \cot^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \cot^{-1}\frac{1}{\tan\theta} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$$

27. Prove that $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Ans:

Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$

We have $\cos(x-y) = \cos x\cos y + \sin x\sin y = \frac{4}{5}\times\frac{15}{17}+\frac{3}{5}\times\frac{8}{17} = \frac{60}{85}+\frac{24}{85} = \frac{84}{85}$

$\Rightarrow x-y = \cos^{-1}\frac{84}{85}$

$\Rightarrow \sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

28. Prove that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16} = \pi$

Ans:

Let $\sin^{-1}\frac{12}{13} = x$, $\cos^{-1}\frac{4}{5} = y$ and $\tan^{-1}\frac{63}{16} = z$

Then $\sin x = \frac{12}{13}$, $\cos y = \frac{4}{5}$ and $\tan z = \frac{63}{16}$

Now, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$

and $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$$\Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5} \quad and \quad \tan y = \frac{\sin y}{\cos y} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x.\tan y} = \frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}.\frac{3}{4}} = \frac{\frac{48+15}{20}}{1-\frac{36}{20}} = \frac{\frac{63}{20}}{-\frac{16}{20}} = -\frac{63}{16} = -\tan z$$

$$\Rightarrow \tan(x+y) = -\tan z = \tan(-z) = \tan(\pi - z)$$

$$\Rightarrow x + y = \pi - z$$

$$\Rightarrow x + y + z = \pi$$

$$\Rightarrow \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$$

29. Simplify: $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), if\ \frac{a}{b}\tan x > -1$

Ans:

$$\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$$

30. Solve: $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

Ans:

Given $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right) = \frac{\pi}{4} \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow (6x-1)(x+1) = 0$$

$$\Rightarrow x = \frac{1}{6} \quad or \quad x = -1$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x = \frac{1}{6}$ is the only solution of the given equation.