

CHAPTER – 3: MATRICES

MARKS WEIGHTAGE – 03 marks

NCERT Important Questions & Answers

1. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

Ans:

Since, a matrix containing 18 elements can have any one of the following orders :

$$1 \times 18, 18 \times 1, 2 \times 9, 9 \times 2, 3 \times 6, 6 \times 3$$

Similarly, a matrix containing 5 elements can have order 1×5 or 5×1 .

2. Construct a 3×4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2} |-3i + j|$ (ii) $a_{ij} = 2i - j$

Ans:

(i) The order of given matrix is 3×4 , so the required matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}, \text{ where } a_{ij} = \frac{1}{2} |-3i + j|$$

Putting the values in place of i and j , we will find all the elements of matrix A .

$$a_{11} = \frac{1}{2} |-3+1| = 1, \quad a_{12} = \frac{1}{2} |-3+2| = \frac{1}{2}, \quad a_{13} = \frac{1}{2} |-3+3| = 0$$

$$a_{14} = \frac{1}{2} |-3+4| = \frac{1}{2}, \quad a_{21} = \frac{1}{2} |-6+1| = \frac{5}{2}, \quad a_{22} = \frac{1}{2} |-6+2| = 2$$

$$a_{23} = \frac{1}{2} |-6+3| = \frac{3}{2}, \quad a_{24} = \frac{1}{2} |-6+4| = 1, \quad a_{31} = \frac{1}{2} |-9+1| = 4$$

$$a_{32} = \frac{1}{2} |-9+2| = \frac{7}{2}, \quad a_{33} = \frac{1}{2} |-9+3| = 3, \quad a_{34} = \frac{1}{2} |-9+4| = \frac{5}{2}$$

Hence, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}_{3 \times 4}$

(ii) Here, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$, where $a_{ij} = 2i - j$

$$\begin{aligned} a_{11} &= 2 - 1 = 1, & a_{12} &= 2 - 2 = 0, \\ a_{13} &= 2 - 3 = -1, & a_{14} &= 2 - 4 = -2, \\ a_{21} &= 4 - 1 = 3, & a_{22} &= 4 - 2 = 2, \\ a_{23} &= 4 - 3 = 1, & a_{24} &= 4 - 4 = 0, \\ a_{31} &= 6 - 1 = 5, & a_{32} &= 6 - 2 = 4, \\ a_{33} &= 6 - 3 = 3 \text{ and } & a_{34} &= 6 - 4 = 2 \end{aligned}$$

Hence, the required matrix is $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}_{3 \times 4}$

3. Find the value of a, b, c and d from the equation: $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Ans:

Given that $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$a - b = -1 \quad \dots(i)$$

$$2a - b = 0 \quad \dots(ii)$$

$$2a + c = 5 \quad \dots(iii)$$

$$\text{and } 3c + d = 13 \quad \dots(iv)$$

Subtracting Eq.(i) from Eq.(ii), we get $a = 1$

Putting $a = 1$ in Eq. (i) and Eq. (iii), we get

$$1 - b = -1 \text{ and } 2 + c = 5$$

$$\Rightarrow b = 2 \text{ and } c = 3$$

Substituting $c = 3$ in Eq. (iv), we obtain

$$3 \times 3 + d = 13 \Rightarrow d = 13 - 9 = 4$$

Hence, $a = 1, b = 2, c = 3$ and $d = 4$.

4. Find X and Y , if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Ans:

$$(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$\text{Now, } (X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} \Rightarrow Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

5. Find the values of x and y from the following equation:

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Ans:

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $2x + 3 = 7$ and $2y - 4 = 14$

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

6. Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Ans:

We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = O$

Ans:

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$\text{So, } A^3 = A.A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$\begin{aligned} \text{Now, } A^3 - 23A - 40I &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} \\ &= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

8. If $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y .

Ans:

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x-y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$2x - y = 10 \dots(i)$$

$$\text{and } 3x + y = 5 \dots(ii)$$

Adding Eqs. (i) and (ii), we get

$$5x = 15 \Rightarrow x = 3$$

Substituting $x = 3$ in Eq. (i), we get

$$2 \times 3 - y = 10 \Rightarrow y = 6 - 10 = -4$$

9. Given $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w .

Ans:

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$3x = x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2$$

$$\text{and } 3y = 6 + x + y \Rightarrow 2y = 6 + x \Rightarrow y = \frac{6+x}{2}$$

Putting the value of x , we get

$$y = \frac{6+2}{2} = \frac{8}{2} = 4$$

Now, $3z = -1 + z + w$, $2z = -1 + w$

$$z = \frac{-1+w}{2} \dots(i)$$

$$\text{Now, } 3w = 2w + 3 \Rightarrow w = 3$$

Putting the value of w in Eq. (i), we get

$$z = \frac{-1+3}{2} = \frac{2}{2} = 1$$

Hence, the values of x, y, z and w are 2, 4, 1 and 3.

10. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x) F(y) = F(x+y)$.

Ans:

$$LHS = F(x)F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\sin y \cos x - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y) = RHS$$

11. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Ans:

$$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2 - 5A + 6I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

12. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

Ans:

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

13. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$

Ans:

Given that $A^2 = kA - 2I$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = k \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$3k - 2 = 1 \Rightarrow k = 1$$

$$-2k = -2 \Rightarrow k = 1$$

$$4k = 4 \Rightarrow k = 1$$

$$-4 = -2k - 2 \Rightarrow k = 1$$

Hence, $k = 1$

14. If $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Ans:

Let $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$ where $x = \tan \frac{\alpha}{2}$

Now, $\cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} = \frac{1 - x^2}{1 + x^2}$ and $\sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} = \frac{2x}{1 + x^2}$

$$RHS = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1-x^2}{1+x^2} & -\frac{2x}{1+x^2} \\ \frac{2x}{1+x^2} & \frac{1-x^2}{1+x^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix} \begin{bmatrix} \frac{1-x^2}{1+x^2} & -\frac{2x}{1+x^2} \\ \frac{2x}{1+x^2} & \frac{1-x^2}{1+x^2} \end{bmatrix} = \begin{bmatrix} \frac{1-x^2+2x^2}{1+x^2} & \frac{-2x+x(1-x^2)}{1+x^2} \\ \frac{-x(1-x^2)+2x}{1+x^2} & \frac{2x^2+1-x^2}{1+x^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+x^2}{1+x^2} & \frac{-2x+x-x^3}{1+x^2} \\ \frac{-x+x^3+2x}{1+x^2} & \frac{1+x^2}{1+x^2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-x-x^3}{1+x^2} \\ \frac{x^3+x}{1+x^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-x(1+x^2)}{1+x^2} \\ \frac{x(x^2+1)}{1+x^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix} = RHS$$

15. Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric

matrix.

Ans:

$$B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \Rightarrow B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(B+B') = \frac{1}{2} \begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix}$$

$$\text{Now } P' = \begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B+B')$ is a symmetric matrix.

$$\text{Also, let } Q = \frac{1}{2}(B-B') = \frac{1}{2} \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & 3 \\ 5/2 & -3 & 0 \end{bmatrix}$$

$$\text{Now } Q' = \begin{bmatrix} 0 & 1/2 & 5/2 \\ -1/2 & 0 & -3 \\ -5/2 & 3 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(B-B')$ is a skew symmetric matrix.

$$\text{Now, } P+Q = \begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & 3 \\ 5/2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

16. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

$$(i) \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \quad (ii) \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \quad (iii) \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \quad (iv) \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$$

Ans:

(i)

$$\text{Let } A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \text{ then } A = P+Q$$

$$\text{where, } P = \frac{1}{2}(A+A') \text{ and } Q = \frac{1}{2}(A-A')$$

$$\text{Now, } P = \frac{1}{2}(A+A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

(ii)

$$\text{Let } A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \text{ then } A = P + Q$$

where, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$$

(iii)

$$\text{Let } A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}, \text{ then } A = P + Q$$

where, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 2 \\ -3/2 & -2 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & 2 \\ 3/2 & -2 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 2 \\ -3/2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = A$$

(iv)

Let $A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$, then $A = P + Q$

where, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2} \left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2} \left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P+Q = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} = A$$

17. If $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ and $A + A' = I$, find the value of α .

Ans:

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \Rightarrow A' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

Now, $A + A' = I$

$$\Rightarrow \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} + \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos \alpha & 0 \\ 0 & 2\cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing the corresponding elements of the above matrices, we have

$$2\cos \alpha = 1 \Rightarrow \cos \alpha = \frac{1}{2} = \cos \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$$

18. Obtain the inverse of the following matrix using elementary operations: $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Ans:

$$\text{Write } A = I A, \text{ i.e., } \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \rightarrow R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \rightarrow R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \rightarrow R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_3 \rightarrow \frac{1}{2} R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_1 \rightarrow R_1 + R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_2 \rightarrow R_2 - 2R_3)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

19. Using elementary transformations, find the inverse of $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$, if it exists.

Ans:

$$\text{Let } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{-1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} A \quad \left(\text{Using } R_1 \rightarrow \frac{1}{6} R_1 \right)$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} A \quad (\text{Using } R_2 \rightarrow R_2 + 2R_1)$$

Now, in the above equation, we can see all the elements are zero in the second row of the matrix on the LHS. Therefore, A^{-1} does not exist.

Note Suppose $A = IA$, after applying the elementary transformation, if any row or column of a matrix on LHS is zero, then A^{-1} does not exist.

20. Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Ans:

We suppose that A is a symmetric matrix, then $A' = A$

$$\text{Consider } (B'AB)' = \{B'(AB)\}' = (AB)'(B')' \quad [\because (AB)' = B'A']$$

$$= B'A'(B) \quad [\because (B')' = B]$$

$$= B'(A'B) = B'(AB) \quad [\because A' = A]$$

$$\Rightarrow (B'AB)' = B'AB$$

which shows that $B'AB$ is a symmetric matrix.

Now, we suppose that A is a skew-symmetric matrix.

Then, $A' = -A$

$$\text{Consider } (B'AB)' = [B'(AB)]' = (AB)'(B')' \quad [\because (AB)' = B'A' \text{ and } (A')' = A]$$

$$= (B' A')B = B' (-A)B = -B' AB \quad [\because A' = -A]$$

$$\Rightarrow (B' AB)' = -B' AB$$

which shows that $B' AB$ is a skew-symmetric matrix.

21. If A and B are symmetric matrices, prove that $AB - BA$ is a skew-symmetric matrix.

Ans:

Here, A and B are symmetric matrices, then $A' = A$ and $B' = B$

Now, $(AB - BA)' = (AB)' - (BA)' \quad (\because (A - B)' = A' - B'$ and $(AB)' = B' A')$

$$= B' A' - A' B' = BA - AB \quad (\because B' = B \text{ and } A' = A)$$

$$= -(AB - BA)$$

$$\Rightarrow (AB - BA)' = -(AB - BA)$$

Thus, $(AB - BA)$ is a skew-symmetric matrix.

22. Using elementary transformations, find the inverse of $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$, if it exists.

Ans:

$$\text{Let } A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}. \text{ We know that } A = IA$$

$$\therefore \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{Using } R_1 \rightarrow R_1 + R_2 - R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -5 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -3 & -3 & 4 \end{bmatrix} A \quad (\text{Using } R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -5 & -10 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 4 \\ -2 & -1 & 2 \end{bmatrix} A \quad (\text{Using } R_2 \leftrightarrow R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{3}{5} & \frac{3}{5} & \frac{-4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A \quad (\text{Using } R_2 \rightarrow -\frac{1}{5}R_2 \text{ and } R_3 \rightarrow -\frac{1}{5}R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A \quad (\text{Using } R_2 \rightarrow R_2 - 2R_3 \text{ and } R_1 \rightarrow R_1 - 4R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{5} & 0 & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A \quad (\text{Using } R_1 \rightarrow R_1 - R_2)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{-2}{5} & 0 & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

23. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

Ans:

We are required to prove that for all $n \in N$

$$P(n) = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$\text{Let } n = 1, \text{ then } P(1) = \begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \dots\dots(i)$$

which is true for $n = 1$.

Let the result be true for $n = k$.

$$P(k) = A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \dots\dots(ii)$$

Let $n = k + 1$

$$P(k+1) = A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix} = \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$\text{Now, LHS} = A^{k+1} = A^k A^1 = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2k).3 + (-4k).1 & (1+2k).(-4) + (-4k).(-1) \\ k.3 + (1-2k).1 & k.(-4) + (1-2k).(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & -1-2(k+1) \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$ whenever it is true for $n = k$. So, by principle of mathematical induction, it is true for all $n \in N$.

24. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

Ans:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

Since Matrix multiplication is associative, therefore $[1 \ 2 \ 1] \begin{bmatrix} 0+4+0 \\ 0+0+x \\ 0+0+2x \end{bmatrix} = O$

$$\Rightarrow [1 \ 2 \ 1] \begin{bmatrix} 4 \\ x \\ 2x \end{bmatrix} = O$$

$$\Rightarrow [4+2x+2x] = O \Rightarrow [4+4x] = O$$

$$\Rightarrow 4+4x=0 \Rightarrow 4x=-4 \Rightarrow x=-1$$

25. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$.

Ans:

Given that $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

26. Find x , if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$?

Ans:

Given that $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

Since Matrix multiplication is associative, therefore $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} x+0+2 \\ 0+8+1 \\ 2x+0+3 \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} x+2 \\ 9 \\ 2x+3 \end{bmatrix} = O$$

$$\Rightarrow [x(x+2) + (-5).9 + (-1)(2x+3)] = O$$

$$\Rightarrow [x^2 - 48] = O \Rightarrow x^2 - 48 = 0 \Rightarrow x^2 = 48$$

$$\Rightarrow x = \pm\sqrt{48} = \pm 4\sqrt{3}$$

27. Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Ans:

Given that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

The matrix given on the RHS of the equation is a 2×3 matrix and the one given on the LHS of the equation is a 2×3 matrix. Therefore, X has to be a 2×2 matrix.

Now, let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\begin{bmatrix} a+4c & 2a+5c & 3a+6c \\ b+4d & 2b+5d & 3b+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Equating the corresponding elements of the two matrices, we have

$$a + 4c = -7, \quad 2a + 5c = -8, \quad 3a + 6c = -9$$

$$b + 4d = 2, \quad 2b + 5d = 4, \quad 3b + 6d = 6$$

$$\text{Now, } a + 4c = -7 \Rightarrow a = -7 - 4c$$

$$2a + 5c = -8 \Rightarrow -14 - 8c + 5c = -8$$

$$\Rightarrow -3c = 6 \Rightarrow c = -2$$

$$\therefore a = -7 - 4(-2) = -7 + 8 = 1$$

$$\text{Now, } b + 4d = 2 \Rightarrow b = 2 - 4d \text{ and } 2b + 5d = 4 \Rightarrow 4 - 8d + 5d = 4$$

$$\therefore -3d = 0 \Rightarrow d = 0$$

$$\therefore b = 2 - 4(0) = 2$$

$$\text{Thus, } a = 1, b = 2, c = -2, d = 0$$

Hence, the required matrix X is $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

28. If $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, **then prove that** $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in N$

Ans:

We shall prove the result by using principle of mathematical induction.

We have $P(n)$: If $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in N$

Let $n = 1$, then $P(1) = A^1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k) = A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

i.e. $P(k + 1) = A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$

$$\text{Now, } P(k + 1) = A^{k+1} = A^k \cdot A = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos k\theta - \sin \theta \sin k\theta & \cos \theta \sin k\theta + \sin \theta \cos k\theta \\ -\sin \theta \cos k\theta + \cos \theta \sin k\theta & -\sin \theta \sin k\theta + \cos \theta \cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta + \theta) & \sin(k\theta + \theta) \\ -\sin(k\theta + \theta) & \cos(k\theta + \theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in N$$

29. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that $CD - AB = O$.

Ans:

Since A, B, C are all square matrices of order 2, and $CD - AB$ is well defined, D must be a square matrix of order 2.

$$\text{Let } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } CD - AB = 0 \Rightarrow \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \dots (1)$$

$$3a + 8c - 43 = 0 \quad \dots (2)$$

$$2b + 5d = 0 \quad \dots (3)$$

$$\text{and } 3b + 8d - 22 = 0 \quad \dots (4)$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

$$\text{Therefore } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

30. By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Ans:

In order to use elementary row operations we may write $A = IA$.

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A, \text{ then } \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \quad (\text{applying } R_2 \rightarrow R_2 - 2R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \quad (\text{applying } R_2 \rightarrow -\frac{1}{5}R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \quad (\text{applying } R_1 \rightarrow R_1 - 2R_2)$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$