CHAPTER – 13: PROBABILITY

MARKS WEIGHTAGE – 10 marks

NCERT Important Questions & Answers

1. **A die is thrown three times. Events A and B are defined as below: A : 4 on the third throw; B : 6 on the first and 5 on the second throw. Find the probability of A given that B has already occurred.**
Ans:
The sample space has 216 outcomes.
Now A =
(1,1,4) (1,2,4) ... (1,6,4) (2,1,4) (2,2,4) ... (2,6,4)
(3,1,4) (3,2,4) ... (3,6,4) (4,1,4) (4,2,4) ...(4,6,4)
(5,1,4) (5,2,4) ... (5,6,4) (6,1,4) (6,2,4) ...(6,6,4)
B = {(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)}
and A ∩ B = {(6,5,4)}.
Now, $P(B) = \frac{6}{216}$ *and* $P(A \cap B) = \frac{1}{216}$
Then $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{216}}{\frac{6}{216}} = \frac{1}{6}$

2. **A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?**
Ans:
Let E be the event that 'number 4 appears at least once' and F be the event that 'the sum of the numbers appearing is 6'.
Then, E = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)}
and F = {(1,5), (2,4), (3,3), (4,2), (5,1)}
We have $P(E) = \frac{11}{36}$ *and* $P(E) = \frac{5}{36}$
Also E∩F = {(2,4), (4,2)}
Therefore $P(E \cap F) = \frac{2}{36}$
Hence, the required probability, $P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$

3. **A black and a red dice are rolled. (a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5. (b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.**
Ans:
Let the first observation be from the black die and second from the red die.
When two dice (one black and another red) are rolled, the sample space
$S = 6 \times 6 = 36$ (equally likely sample events)
(i) Let E : set of events in which sum greater than 9 and F : set of events in which black die resulted in a 5
E = {(6,4), (4,6), (5, 5), (5,6), (6, 5), (6,6)} $\Rightarrow n(E) = 6$
and F = {(5, 1), (5,2), (5, 3), (5, 4), (5, 5), (5,6)} $\Rightarrow n(F) = 6$
$\Rightarrow E \cap F$ = {(5, 5), (5,6)} $\Rightarrow n(E \cap F) = 2$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by P(E/F)

$$P(E)=\frac{6}{36} \text{ and } P(F)=\frac{6}{36}$$

Also, $P(E\cap F)=\frac{2}{36}$

$$\therefore\ P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{2}{36}}{\frac{6}{36}}=\frac{2}{6}=\frac{1}{3}$$

(ii) Let E : set of events having 8 as the sum of the observations,

F : set of events in which red die resulted in a (in any one die) number less than 4

$\Rightarrow E$ = {(2,6), (3, 5), (4,4), (5, 3), (6,2)} $\Rightarrow n(E) = 5$

and F = {(1, 1), (1, 2),….. (3, 1), (3, 2)…… (5, 1), (5, 2),………} $\Rightarrow n(F) = 18$

$\Rightarrow E \cap F$ = {(5, 3), (6, 2)} $\Rightarrow n(E \cap F) = 2$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by P(E/F)

$$P(E)=\frac{5}{36} \text{ and } P(F)=\frac{18}{36}$$

Also, $P(E\cap F)=\frac{2}{36}$

$$\therefore\ P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{2}{36}}{\frac{18}{36}}=\frac{2}{18}=\frac{1}{9}$$

4. **An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?**

Ans:

Total number of questions = 300 + 200 + 500 + 400 = 1400.

Let E be the event that selected question is an easy question

Then, $n(E)$ = 500 + 300 = 800

$$\therefore\ P(E)=\frac{800}{1400}$$

Let F be the event that selected question is a multiple choice question.

Then, $n(F)$ = 500 + 400 = 900

$$\therefore P(F)=\frac{900}{1400}$$

Also, $P(E\cap F)=\frac{500}{1400}$

$$\Rightarrow P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{500}{1400}}{\frac{900}{1400}}=\frac{500}{900}=\frac{5}{9}$$

5. **An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?**

Ans:

Let E and F denote respectively the events that first and second ball drawn are black. We have to find P(E ∩ F) or P (EF).

Now P(E) = P (black ball in first draw) = $\frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. P(F|E) = $\frac{9}{14}$

By multiplication rule of probability, we have

P (E ∩ F) = P(E) P(F|E) = $\frac{10}{15}\times\frac{9}{14}=\frac{3}{7}$

6. **Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (i) both balls are red. (ii) first ball is black and second is red. (iii) one of them is black and other is red.**

Ans:

Total number of balls = 18, number of red balls = 8 and number of black balls = 10

∴ Probability of drawing a red ball = $\frac{8}{18}$

Similarly, probability of drawing a black ball = $\frac{10}{18}$

(i) Probability of getting both red balls = P (both balls are red)

= P (a red ball is drawn at first draw and again a red ball at second draw) = $\frac{8}{18}\times\frac{8}{18}=\frac{16}{81}$

(ii) P (probability of getting first ball is black and second is red) = $\frac{10}{18}\times\frac{8}{18}=\frac{20}{81}$

(iii) Probability of getting one black and other red ball = P(first ball is black and second is red) + P (first ball is red and second is black) = $\frac{10}{18}\times\frac{8}{18}+\frac{8}{18}\times\frac{10}{18}=\frac{20}{81}+\frac{20}{81}=\frac{40}{81}$

7. **Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem.**

Ans:

Probability of solving the problem by A, $P(A)$ = $\frac{1}{2}$

Probability of solving the problem by B, $P(B)$ = $\frac{1}{3}$

Probability of not solving the problem by A = $P(A')$ = 1 – $P(A)$ = $1-\frac{1}{2}=\frac{1}{2}$

and probability of not solving the problem by B = P(B') = 1 – P(B) = $1-\frac{1}{3}=\frac{2}{3}$

(i) P (the problem is solved) = 1 – P(none of them solve the problem) =

$=1-P(A'\cap B')=1-P(A')P(B')$

(since A and B are independent A' and B' are independent)

$=1-\left(\frac{1}{2}\times\frac{2}{3}\right)=1-\frac{1}{3}=\frac{2}{3}$

(ii) P (exactly one of them solve the problem) = $P(A)\,P(B') + P(A')\,P(B)$

$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}=\frac{1}{3}+\frac{1}{6}=\frac{2+1}{6}=\frac{3}{6}=\frac{1}{2}$

8. In a hostel, 60% of the students read Hindi news paper, 40% read English news paper and 20% read both Hindi and English news papers. A student is selected at random. (a) Find the probability that she reads neither Hindi nor English news papers. (b) If she reads Hindi news paper, find the probability that she reads English news paper. (c) If she reads English news paper, find the probability that she reads Hindi news paper.

Ans:

Let H : Set of students reading Hindi newspaper and E : set of students reading English newspaper.

Let $n(S) = 100$ Then, $n(H) = 60$

$n(E) = 40$ and $n(H \cap E) = 20$

$$\therefore P(H) = \frac{60}{100} = \frac{3}{5}, P(E) = \frac{40}{100} = \frac{2}{5} \text{ and } P(H \cap E) = \frac{20}{100} = \frac{1}{5}$$

(i) Required probability = P (student reads neither Hindi nor English newspaper) =

$$= P(H' \cap E') = P(H \cup E)' = 1 - P(H \cup E)$$

$$= 1 - [P(H) + P(E) - P(H \cap E)] = 1 - \left[\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right] = 1 - \frac{4}{5} = \frac{1}{5}$$

(ii) Required probability = P(a randomly chosen student reads English newspaper, if he/she reads Hindi newspaper) = $P(E/H) = \dfrac{P(E \cap H)}{P(H)} = \dfrac{\frac{1}{5}}{\frac{3}{5}} = \dfrac{1}{3}$

(iii) Required probability = P (student reads Hindi newspaper when it is given that reads English newspaper) = $P(H/E) = \dfrac{P(H \cap E)}{P(E)} = \dfrac{\frac{1}{5}}{\frac{2}{5}} = \dfrac{1}{2}$

9. Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Ans:

Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $P(E_1) = P(E_2) = \dfrac{1}{2}$

Also $P(A|E_1)$ = P(drawing a red ball from Bag I) = $\dfrac{3}{7}$

and $P(A|E_2)$ = P(drawing a red ball from Bag II) = $\dfrac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2 / A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} = \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{35}{68}$$

10. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Ans:

Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1)$ = P(a gold coin from bag I) = 2/2 =1

$P(A|E_2)$ = P(a gold coin from bag II) = 0

$P(A|E_3)$ = P(a gold coin from bag III) = $\frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

= $P(E_1|A)$

By Bayes' theorem, we know that

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times 0+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

11. In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Ans:

Let events B_1, B_2, B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$P(B_1) = 25\% = 0.25$, $P(B_2) = 0.35$ and $P(B_3) = 0.40$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05

Similarly, $P(E|B_2) = 0.04$, $P(E|B_3) = 0.02$.

Hence, by Bayes' Theorem, we have

$$P(B_2/E) = \frac{P(B_2)P(E/B_2)}{P(B_1)P(E/B_1)+P(B_2)P(E/B_2)+P(B_3)P(E/B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05+0.35\times 0.04+0.40\times 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}$$

12. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4},\frac{1}{3}$ *and* $\frac{1}{2}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Ans:

Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then $P(T_1) = \frac{3}{10}$, $P(T_2) = \frac{1}{5}$, $P(T_3) = \frac{1}{10}$ and $P(T_4) = \frac{2}{5}$

$P(E|T_1)$ = Probability that the doctor arriving late comes by train = $\frac{1}{4}$

Similarly, $P(E|T_2) = \frac{1}{3}$, $P(E|T_3) = \frac{1}{12}$ and $P(E|T_4) = 0$, since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = Probability that the doctor arriving late comes by train

$$\Rightarrow P(T_1/E) = \frac{P(T_1)P(E/T_1)}{P(T_1)P(E/T_1) + P(T_2)P(E/T_2) + P(T_3)P(E/T_3) + P(T_4)P(E/T_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$

13. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Ans:

Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $P(S_1)$ = Probability that *six* occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur = $\frac{5}{6}$

$P(E|S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth = $\frac{3}{4}$

$P(E|S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth = $1-\frac{3}{4}=\frac{1}{4}$

Thus, by Bayes' theorem, we get

$P(S_1|E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$\Rightarrow P(S_1/E) = \frac{P(S_1)P(E/S_1)}{P(S_1)P(E/S_1)+P(S_2)P(E/S_2)} = \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{1}{8}\times\frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is $\frac{3}{8}$.

14. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Ans:

Let E_1 : first bag is selected, E_2 : second bag is selected

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1) = P(E_2) = \frac{1}{2}$

Let E : ball drawn is red.

$P(E/E_1)$ = P(drawing a red ball from first bag) = $\frac{4}{8}=\frac{1}{2}$

$P(E/E_2)$ = P(drawing a red ball from second bag) = $\frac{2}{8}=\frac{1}{4}$

By using Baye's theorem,

Required probability = $P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}}=\frac{\frac{1}{4}}{\frac{2+1}{8}}=\frac{\frac{1}{4}}{\frac{3}{8}}=\frac{2}{3}$$

15. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Ans:

Let E_1 : the event that the student is residing in hostel and E_2 : the event that the student is not residing in the hostel.

Let E : a student attains A grade,

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1)$ = 60% = $\frac{60}{100}=\frac{3}{5}$ and $P(E_2)$ = 40% = $\frac{40}{100}=\frac{2}{5}$

Then $P(E/E_1)$ = 30% = $\frac{30}{100}=\frac{3}{10}$ and $P(E/E_2)$ = 20% = $\frac{20}{100}=\frac{2}{10}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}=\frac{\frac{3}{5}\times\frac{3}{10}}{\frac{3}{5}\times\frac{3}{10}+\frac{2}{5}\times\frac{2}{10}}=\frac{9}{9+4}=\frac{9}{13}$$

16. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Ans:

Let E_1 : the event that the student knows the answer and E_2 : the event that the student guesses the answer.

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1)$ = $\frac{3}{4}$ and $P(E_2)$ = $\frac{1}{4}$

Let E : the answer is correct.

The probability that the student answered correctly, given that he knows the answer, is 1 *i.e., P* $P(E/E_1) = 1$

Probability that the students answered correctly, given that the he guessed, is $\frac{1}{4}$

i.e., $P(E/E_2) = \frac{1}{4}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{3}{4}\times 1}{\frac{3}{4}\times 1+\frac{1}{4}\times\frac{1}{4}}=\frac{\frac{3}{4}}{\frac{3}{4}+\frac{1}{16}}=\frac{\frac{12}{16}}{\frac{12+1}{16}}=\frac{12}{13}$$

17. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Ans:

Let E_1 : the event that the coin chosen is two headed, E_2 : the event that the coin chosen is biased and E_3 : the event that the coin chosen is unbiased

$\Rightarrow E_1, E_2, E_3$ are mutually exclusive and exhaustive events. Moreover,

$\Rightarrow P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Let E : tosses coin shows up a head,

$\therefore$ $P(E/E_1)$ = P(coin showing heads, given that it is a two headed coin) = 1

$P(E/E_2)$ = P(coin showing heads, given that it is a biased coin) = $75\%=\frac{75}{100}=\frac{3}{4}$

$P(E/E3)$ = P(coin showing heads, given that it is an unbiased coin) = $\frac{1}{2}$

The probability that the coin is two headed, given that it shows head, is given by $P(E_1/E)$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}}=\frac{1}{1+\frac{3}{4}+\frac{1}{2}}=\frac{1}{\frac{4+3+2}{4}}=\frac{4}{9}$$

18. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Ans:

There are 2000 scooter drivers, 4000 car drivers and 6000 truck drivers.

Total number of drivers = 2000 + 4000 + 6000 = 12000

Let E_1 : the event that insured person is a scooter driver, E_2 : the event that insured person is a car driver and E_3 : the event that insured person is a truck driver.

Then, E_1, E_2, E_3 are mutually exclusive and exhaustive events. Moreover,

P(E_1) = $\frac{2000}{12000}=\frac{1}{6}$, P($E_2$) = $\frac{4000}{12000}=\frac{1}{3}$ and P(E_3) = $\frac{6000}{12000}=\frac{1}{2}$

Let E : the events that insured person meets with an accident,

$\therefore$ $P(E/E_1)$ = P(scooter driver met with an accident) = $0.01=\frac{1}{100}$

$P(E/E_2)$ = P(car driver met with an accident) = $0.03=\frac{3}{100}$

$P(E/E3)$ = P(truck driver met with an accident) = $0.15=\frac{15}{100}$

The probability that the driver is a scooter driver, given he met with an accident, is given by $P(E_1/E)$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{1}{100}\times\frac{1}{6}}{\frac{1}{100}\times\frac{1}{6}+\frac{3}{100}\times\frac{1}{3}+\frac{15}{100}\times\frac{1}{2}}=\frac{\frac{1}{6}}{\frac{1}{6}+1+\frac{15}{2}}=\frac{1}{1+6+45}=\frac{1}{52}$$

19. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

Ans:

Let E_1 : the event that the item is produced by machine A and E_2 : the event that the item is produced by machine B.

Then, E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$P(E_1)$ = 60% $=\frac{60}{100}=\frac{3}{5}$ and $P(E_2)$ = 40% = $\frac{40}{100}=\frac{2}{5}$

Let E : the event that the item chosen is defective,

$\therefore$ $P(E/E_1)$ = P(machine A produced defective items) = 2% $=\frac{2}{100}$

$P(E/E_2)$ = P(machine B produced defective items) = 1% $=\frac{1}{100}$

The probability that the randomly selected item was from machine B, given that it is defective, is given by $P(E_2/E)$

By using Baye's theorem, we obtain

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{2}{5}\times\frac{1}{100}}{\frac{3}{5}\times\frac{2}{100}+\frac{2}{5}\times\frac{1}{100}}=\frac{\frac{2}{500}}{\frac{6}{500}+\frac{2}{500}}=\frac{2}{6+2}=\frac{2}{8}=\frac{1}{4}$$

20. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Ans:

Let E_1 : the event that 5 or 6 is shown on die and E_2 : the event that 1, 2, 3, or 4 is shown on die.

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

and $n(E_1)$ = 2, $n(E_2)$ = 4

Also, $n(S)$ = 6

$P(E_1)$ = $\frac{2}{6}=\frac{1}{3}$ and $P(E_2)$ = $\frac{4}{6}=\frac{2}{3}$

Let E : The event that exactly one head show up,

$\therefore$ $P(E/E_1)$ = P(exactly one head show up when coin is tossed thrice) = P{HTT, THT, TTH} = $\frac{3}{8}$

$P(E/E_2)$ = P(head shows up when coin is tossed once) = $\frac{1}{2}$

The probability that the girl threw, 1, 2, 3 or 4 with the die, if she obtained exactly one head, is given by $P(E_2/E)$

By using Baye's theorem, we obtain

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}}=\frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}}=\frac{8}{3+8}=\frac{8}{11}$$

21. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

Ans:

Let E_1 : the event that item is produced by machine A, E_2 : the event that item is produced by machine B and E_3 : the event that item is produced by machine C

Here, E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

Moreover,

$P(E_1)$ = 50% $=\frac{50}{100}$

$P(E_2)$ = 30% = $\frac{30}{100}$

and $P(E_3)$ = 20% = $\frac{20}{100}$

Let E : The event that item chosen is found to be defective',

$\therefore P(E/E_1)=\frac{1}{100}$, $P(E/E_2)=\frac{5}{100}$, $P(E/E_3)=\frac{7}{100}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{50}{100}\times\frac{1}{100}}{\frac{50}{100}\times\frac{1}{100}+\frac{30}{100}\times\frac{5}{100}+\frac{20}{100}\times\frac{7}{100}}=\frac{50}{50+150+140}=\frac{50}{340}=\frac{5}{34}$$

22. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

Ans:

Let E_1 : the event that lost cards is a diamond $\Rightarrow n(E_1)=13$

E_2 : lost cards is not a diamond $\Rightarrow n(E_2)=52-13=39$

And, $n(S)=52$

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

$\therefore P(E_1)=\frac{13}{52}=\frac{1}{4}$ and $P(E_2)=\frac{39}{52}=\frac{3}{4}$

Let E : the events that two cards drawn from the remaining pack are diamonds,

When one diamond card is lost, there are 12 diamond cards out of 51 cards.

The cards can be drawn out of 12 diamond cards in $^{12}C_2$ ways.

Similarly, 2 diamond cards can be drawn out of 51 cards in ${}^{51}C_2$ ways. The probability of getting two cards, when one diamond card is lost, is given by P(E/E_1)

$$\therefore P(E/E_1) = \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{\frac{12\times 11}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{12\times 11}{51\times 50} = \frac{132}{2550}$$

$$\text{and } P(E/E_2) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{\frac{13\times 12}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{13\times 12}{51\times 50} = \frac{156}{2550}$$

By using Baye's theorem, we obtain

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$$

$$= \frac{\frac{1}{4}\times\frac{132}{2550}}{\frac{1}{4}\times\frac{132}{2550} + \frac{3}{4}\times\frac{156}{2550}} = \frac{132}{132+468} = \frac{132}{600} = \frac{11}{50}$$

23. Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Ans:

The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, or 2.

Now, since the draws are done with replacement, therefore, the two draws form independent experiments.

Therefore, P(X = 0) = P(non-ace and non-ace) = P(non-ace) × P(non-ace) $= \frac{48}{52}\times\frac{48}{52} = \frac{144}{169}$

P(X = 1) = P(ace and non-ace or non-ace and ace)

= P(ace and non-ace) + P(non-ace and ace)

= P(ace). P(non-ace) + P (non-ace) . P(ace) $= \frac{4}{52}\times\frac{48}{52} + \frac{48}{52}\times\frac{4}{52} = \frac{24}{169}$

and P(X = 2) = P (ace and ace) $= \frac{4}{52}\times\frac{4}{52} = \frac{1}{169}$

Thus, the required probability distribution is

X	0	1	2
P(X)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

24. Find the probability distribution of number of doublets in three throws of a pair of dice.

Ans:

Let X denote the number of doublets. Possible doublets are

(1,1) , (2,2), (3,3), (4,4), (5,5), (6,6)

Clearly, X can take the value 0, 1, 2, or 3.

Probability of getting a doublet $= \frac{6}{36} = \frac{1}{6}$

Probability of not getting a doublet $= 1 - \frac{1}{6} = \frac{5}{6}$

Now P(X = 0) = P (no doublet) $= \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{125}{216}$

P(X = 1) = P (one doublet and two non-doublets) $= \frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

$= 3\left(\frac{1}{6}\times\frac{5^2}{6^2}\right) = \frac{75}{216}$

P(X = 2) = P (two doublets and one non-doublet) $= \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}$

$= 3\left(\frac{1}{6^2}\times\frac{5}{6}\right) = \frac{15}{216}$

and P(X = 3) = P (three doublets) $= \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} = \frac{1}{216}$

Thus, the required probability distribution is

X	0	1	2	3
P(X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

25. Find the variance of the number obtained on a throw of an unbiased die.

Ans:

The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

Also P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = $\frac{1}{6}$

Therefore, the Probability distribution of X is

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now E(X) = $\sum_{i=1}^{n} x_i p(x_i) = 1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6} = \frac{21}{6}$

Also E(X^2) $= 1^2\times\frac{1}{6}+2^2\times\frac{1}{6}+3^2\times\frac{1}{6}+4^2\times\frac{1}{6}+5^2\times\frac{1}{6}+6^2\times\frac{1}{6} = \frac{91}{6}$

Thus, Var (X) = E (X^2) – (E(X))^2 $= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} = \frac{35}{12}$

26. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Ans:

Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now P(X = 0) = P (no king) $= \frac{^{48}C_2}{^{52}C_2} = \frac{\frac{48!}{2!(48-2)!}}{\frac{52!}{2!(52-2)!}} = \frac{48\times 47}{52\times 52} = \frac{188}{221}$

P(X = 1) = P (one king and one non-king) $= \frac{^{4}C_1\,^{48}C_1}{^{52}C_2} = \frac{4\times 48\times 2}{52\times 51} = \frac{32}{221}$

and P(X = 2) = P (two kings) $= \frac{^4C_2}{^{52}C_2} = \frac{4\times3}{52\times51} = \frac{1}{221}$

Thus, the probability distribution of X is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Now Mean of X = E(X) = $\sum_{i=1}^{n} x_i p(x_i) = 0\times\frac{188}{221} + 1\times\frac{32}{221} + 2\times\frac{1}{221} = \frac{34}{221}$

Also E(X^2) = $0^2\times\frac{188}{221} + 1^2\times\frac{32}{221} + 2^2\times\frac{1}{221} = \frac{36}{221}$

Thus, Var (X) = E (X^2) – (E(X))^2 = $= \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{(221)^2}$

Therefore $\sigma_x = \sqrt{Var(x)} = \sqrt{\frac{6800}{(221)^2}} = \frac{\sqrt{6800}}{221} = 0.37$

27. Find the probability distribution of (i) number of heads in two tosses of a coin. (ii) number of tails in the simultaneous tosses of three coins. (iii) number of heads in four tosses of a coin.

Ans:

(i) When one coin is tossed twice, the sample space is $S = \{HH, HT, TH, TT\}$.

Let X denotes, the number of heads in any outcome in S,

$X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$ and $X(TT) = 0$

Therefore, X can take the value of 0, 1 or 2. It is known that

$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$

$\therefore$ $P(X = 0) = P$ (tail occurs on both tosses) = $P(\{TT\}) = \frac{1}{4}$

$P(X = 1) = P$ (one head and one tail occurs) = $P(\{TH, HT\}) = \frac{2}{4} = \frac{1}{2}$

and $P(X = 2) = P$ (head occurs on both tosses) = $P(\{HH\}) = \frac{1}{4}$

Thus, the required probability distribution is as follows

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii) When three coins are tossed thrice, the sample space is S = $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ which contains eight equally likely sample points.

Let X represent the number of tails. Then, X can take values 0, 1, 2 and 3.

$P(X = 0) = P$ (no tail) = $P(\{HHH\}) = \frac{1}{8}$,

$P(X = 1) = P$ (one tail and two heads show up) = $P(\{HHT, HTH, THH\}) = \frac{3}{8}$,

$P(X = 2) = P$ (two tails and one head show up) = $P(\{HTT, THT, TTH\}) = \frac{3}{8}$

and $P(X = 3) = P$ (three tails show up) = $P(\{TTT\}) = \frac{1}{8}$

Thus, the probability distribution is as follows

X	0	1	2	3
P (X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii) When a coin is tossed four times, the sample space is
S = {HHHH, HHHT, HHTH, HTHT, HTTH, HTTT, THHH, HTHH, THHT, THTH, HHTT, TTHH, TTHT, TTTH, THTT, TTTT} which contains 16 equally likely sample points.
Let *X* be the random variable, which represents the number of heads. It can be seen that *X* can take the value of 0, 1, 2, 3 or 4.

$P(X = 0) = P(\text{no head shows up}) = P\{TTTT\} = \frac{1}{16}$,

$P(X = 1) = P$ (one head and three tails show up) $= P(HTTT, THTT, TTHT, TTTH) = \frac{4}{16} = \frac{1}{4}$,

$P(X = 2) = P$ (two heads and two tails show up) $= P(\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\})$
$= \frac{6}{16} = \frac{3}{8}$,

$P(X = 3) = P$ (three heads and one tail show up) $= P(\{HHHT, HHTH, HTHH, THHH\}) = \frac{4}{16} = \frac{1}{4}$

and $P(X = 4) = P$ (four heads show up) $= P(\{HHHH\}) = \frac{1}{16}$

Thus, the probability distribution is as follows:

X	0	1	2	3	4
P (X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

28. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as (i) number greater than 4 (ii) six appears on at least one die

Ans:

When a die is tossed two times, we obtain (6 x 6) = 36 number of sample points.
(i) Let *X* be the random variable which denotes the number greater than 4 in two tosses of a die. So *X* may have values 0, 1 or 2.

Now, $P(X = 0) = P$ (number less than or equal to 4 on both the tosses) $= \frac{4}{6} \times \frac{4}{6} = \frac{16}{36} = \frac{4}{9}$,

$P(X = 1) = P$ (number less than or equal to 4 on first toss and greater than 4 on second toss) + P(number greater than 4 on first toss and less than or equal to 4 on second toss)
$= \frac{4}{6} \times \frac{2}{6} + \frac{4}{6} \times \frac{2}{6} = \frac{8}{36} + \frac{8}{36} = \frac{16}{36} = \frac{4}{9}$

$P(X = 2) = P$ (number greater than 4 on both the tosses) $= \frac{2}{6} \times \frac{2}{6} = \frac{4}{36} = \frac{1}{9}$

Probability distribution of *X, i.e.,* number of successes is

X	0	1	2
P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii) Let *X* be the random variable which denotes the number of six appears on atleast one die. So, *X* may have values 0 or 1.

$P(X = 0) = P$ (six does not appear on any of the die) $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$P(X = 1) = P$ (six appears on atleast one of the die) = $\frac{11}{36}$

Thus, the required probability distribution is as follows

X	0	1
P(X)	$\frac{25}{36}$	$\frac{11}{36}$

29. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Ans:

It is given that out of 30 bulbs, 6 are defective.

Number of non-defective bulbs = 30 – 6 = 24

4 bulbs are drawn from the lot with replacement.

Let $p = P$(obtaining a defective bulb when a bulb is drawn) = $\frac{6}{30} = \frac{1}{5}$

and $q = P$(obtaining a non-defective bulb when a bulb is drawn) = $\frac{24}{30} = \frac{4}{5}$

Using Binomial distribution, we have

$P(X = 0) = P$ (no defective bulb in the sample) = ${}^4C_0 p^0 q^4 = \left(\frac{4}{5}\right)^4 = \frac{256}{625}$

$P(X = 1) = P$ (one defective bulb in the sample) = ${}^4C_1 p^1 q^3 = 4\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^3 = \frac{256}{625}$

$P(X = 2) = P$ (two defective and two non-defective bulbs are drawn) = ${}^4C_2 p^2 q^2 = 6\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2 = \frac{96}{625}$

$P(X = 3) = P$ (three defective and one non-defective bulbs are drawn) =

${}^4C_3 p^3 q^1 = 4\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^1 = \frac{16}{625}$

$P(X = 4) = P$ (four defective bulbs are drawn) = ${}^4C_4 p^4 q^0 = \left(\frac{1}{5}\right)^4 = \frac{1}{625}$

Therefore, the required probability distribution is as follows.

X	0	1	2	3	4
P (X)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

30. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

Ans:

Let X denotes the random variable which denotes the number of tails when a biased coin is tossed twice.

So, X may have value 0, 1 or 2.

Since, the coin is biased in which head is 3 times as likely to occur as a tail.

$\therefore P\{H\} = \frac{3}{4}$ and $P\{T\} = \frac{1}{4}$

$P(X = 1) = P\{HH\} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$P(X = 1) = P$ (one tail and one head) = $P\{HT,TH\} = P\{HT\} + P\{TH\} + P\{H\}P\{T\} + P\{T\}P\{H\}$

$= \frac{3}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{3}{4}=\frac{3}{16}+\frac{3}{16}=\frac{6}{16}=\frac{3}{8}$

$P(X = 2) = P$ (two tails) $= P\{TT\} = P\{T\}\ P\{T\} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Therefore, the required probability distribution is as follows

X	0	1	2
P(X)	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

31. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine
(i) k (ii) P(X < 3)
(iii) P(X > 6) (iv) P(0 < X < 3)
Ans:
(i) It is known that the sum of a probability distribution of random variable is one *i.e.*, $\sum P(X) = 1$, therefore
$P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$
$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$
$\Rightarrow 10k^2+9k-1=0$
$\Rightarrow (k+1)(10k-1)=0$
$\Rightarrow k=-1 \ \ or \ \ k=\frac{1}{10}$
k = – 1 is not possible as the probability of an event is never negative.
$\therefore k=\frac{1}{10}$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 0+k+2k=3k=\frac{3}{10}$

(iii) $P(X > 6) = P(7) = 7k^2+k=\frac{7}{100}+\frac{1}{10}=\frac{7+10}{100}=\frac{17}{100}$

(iv) $P(0 < X < 3) = P(1) + P(2) = k+2k=3k=\frac{3}{10}$

32. The random variable X has a probability distribution P(X) of the following form, where *k* is some number :

$$P(X)=\begin{cases} k \ \ if \ \ x=0 \\ 2k \ \ if \ \ x=1 \\ 3k, \ \ if \ \ x=2 \\ 0, otherwise \end{cases}$$

(a) Determine the value of *k*.
(b) Find P (X < 2), P (X ≤ 2), P(X ≥ 2).
Ans:
Given distribution of *X* is

X	0	1	2	otherwise
P(X)	k	$2k$	$3k$	0

(a) Since, $\sum P(X) = 1$, therefore $P(0) + P(1) + P(2) + P$ (otherwise) $= 1$

$$\Rightarrow k + 2k + 3k + 0 = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$$

(b) $P(X = 2) = P(0) + P(1) = k + 2k = 3k = 3\times\frac{1}{6} = \frac{1}{2}$

$P(X \le 2) = P(0) + P(1) + P(2) = k + 2k + 3k = 6k = 6\times\frac{1}{6} = 1$

and $P(X \ge 2) = P(2) + P(\text{otherwise}) = 3k + 0 = 3k = 3\times\frac{1}{6} = \frac{1}{2}$

33. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

Ans:

Here, total students = 15

The ages of students in ascending order are 14, 14, 15, 16, 16, 17, 17, 17, 18, 19, 19, 20, 20, 20, 21

Now, $P(X = 14) = \frac{2}{15}$, $P(X = 15) = \frac{1}{15}$, $P(X = 16) = \frac{2}{15}$, $P(X = 17) = \frac{3}{15}$

$P(X = 18) = \frac{1}{15}$, $P(X = 19) = \frac{2}{15}$, $P(X = 20) = \frac{3}{15}$, $P(X = 21) = \frac{1}{15}$

Therefore, the probability distribution of random variable X is as follows :

X	14	15	16	17	18	19	20	21
Number of students	2	1	2	3	1	2	3	1
P(X)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

The third row gives the probability distribution of X.

$Mean\ \ X = \sum XP(X)$

$$= \frac{14\times2+15\times1+16\times2+17\times3+18\times1+19\times2+20\times3+21\times1}{15}$$

$$= \frac{263}{15} = 17.53$$

Variance X = $\sum X^2P(X) - (Mean)^2$

$$= \frac{14^2\times2+15^2\times1+16^2\times2+17^2\times3+18^2\times1+19^2\times2+20^2\times3+21^2\times1}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{392+225+512+867+324+722+1200+441}{15} - 307.4$$

$$= \frac{4683}{15} - 307.4 = 312.2 - 307.4 = 4.8$$

SD of X = $\sqrt{Var} = \sqrt{4.8} = 2.19$

34. If a fair coin is tossed 10 times, find the probability of (i) exactly six heads (ii) at least six heads (iii) at most six heads.

Ans:

The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{1}{2}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$
Here $n = 10$,
$p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore $P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6!\times 4!} \times \frac{1}{2^{10}} = \frac{105}{512}$

(ii) P(at least six heads) = $P(X \geq 6)$
$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^{10} + {}^{10}C_7 \left(\frac{1}{2}\right)^{10} + {}^{10}C_8 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$

$= \left[\left(\frac{10!}{6!\times 4!}\right) + \left(\frac{10!}{7!\times 3!}\right) + \left(\frac{10!}{8!\times 2!}\right) + \left(\frac{10!}{9!\times 1!}\right) + \left(\frac{10!}{10!}\right)\right] \frac{1}{2^{10}} = \frac{193}{512}$

(iii) P(at most six heads) = $P(X \leq 6)$
$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$= {}^{10}C_0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + {}^{10}C_2 \left(\frac{1}{2}\right)^{10} + {}^{10}C_3 \left(\frac{1}{2}\right)^{10} + {}^{10}C_4 \left(\frac{1}{2}\right)^{10} + {}^{10}C_5 \left(\frac{1}{2}\right)^{10} + {}^{10}C_6 \left(\frac{1}{2}\right)^{10}$

$= \frac{848}{1024} = \frac{53}{64}$

35. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of (i) 5 successes? (ii) at least 5 successes? (iii) at most 5 successes?

Ans:

The repeated tosses of a die are Bernoulli trials. Let X denote
the number of successes of getting odd numbers in an experiment of 6
trials.

$p = P$ (success) = P (getting an odd number in a single throwof a die)

$\therefore p = \frac{3}{6} = \frac{1}{2}$ and $q = P$ (failure) = $1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, by Binomial distribution
$P(X = r) = {}^nC_r\, p^{n-r}\, q^r$, where $r = 0, 1, 2,...,n$

$P(X = r) = {}^6C_r \left(\frac{1}{2}\right)^{6-r} \left(\frac{1}{2}\right)^r = {}^6C_r \left(\frac{1}{2}\right)^6$

(i) P (5 successes) = ${}^6C_5 \left(\frac{1}{2}\right)^6 = 6 \times \frac{1}{2^6} = \frac{6}{64} = \frac{3}{32}$

(ii) P (atleast 5 successes) = P (5 successes) + P (6 successes) = ${}^6C_5 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6 = \frac{6}{64} + \frac{1}{64} = \frac{7}{64}$

(iii) P (atmost 5 successes) = 1 – P (6 successes) = $1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}$

36. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Ans:

Let X denote the number of defective items in a sample of 10 items drawn successively. Since, the drawing is done with replacement, the trials are Bernoulli trials.

$p = P$ (success) = 5% = $\frac{5}{100} = \frac{1}{20}$

and $q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

X has a binomial distribution with $n = 10$ and $p = \frac{1}{20}$ and $q = \frac{19}{20}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2, ..., n$

$$P(X = r) = {}^{10}C_r \left(\frac{1}{20}\right)^r \left(\frac{19}{20}\right)^{10-r}$$

Required probability = P (not more than one defective item)

= $P(0) + P(1) = {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9 = q^{10} + 10pq^9$

$$= q^9(q + 10p) = \left(\frac{19}{20}\right)^9 \left(\frac{19}{20} + 10 \times \frac{1}{20}\right) = \frac{29}{20}\left(\frac{19}{20}\right)^9$$

37. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that (i) all the five cards are spades? (ii) only 3 cards are spades? (iii) none is a spade?

Ans:

Let X represent the number of spade cards among the five cards drawn. Since, the drawing card is with replacement, the trials are Bernoulli trials.

In a well-shuffled deck of 52 cards, there are 13 spade cards.

$p = P$ (success) = P (a spade card is drawn) = $\frac{13}{52} = \frac{1}{4}$

and $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

X has a binomial distribution with $n = 5$, $p = \frac{1}{4}$ and $q = \frac{3}{4}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2, ..., n$

$$P(X = r) = {}^5C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{5-r}$$

(i) P (all the five cards are spades) = $P(X = 5) = {}^5C_5 p^5 q^0 = p^5 = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

(ii) P (only three cards are spades) = $P(X = 3) = {}^5C_3 p^3 q^2 = 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{1024} = \frac{45}{512}$

(iii) P (none is a spade) = $P(X = 0) = {}^5C_0 p^0 q^5 = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$

38. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Ans:

The repeated selections of articles in a random sample space are Bernoulli trials. Let X denotes the number of times of selecting defective articles in a random sample space of 12 articles.

Here, p = 10% = $\frac{10}{100} = \frac{1}{10}$

and $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

Clearly, X has a binomial distribution with $n = 12$, $p = \frac{1}{10}$ and $q = \frac{9}{10}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2,...,n$

$$P(X = r) = {}^{12}C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{12-r}$$

Required probability = P (9 items are defective) = $P(X = 9) = {}^{12}C_3 p^9 q^3 = {}^{12}C_3 \left(\frac{1}{10}\right)^9 \left(\frac{9}{10}\right)^3$

$$= \frac{12\times11\times10}{1\times2\times3}.\frac{9^3}{10^{12}} = \frac{22\times9^3}{10^{11}}$$

39. The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Ans:

Let the shooter fire n times. Obviously, n fires are n Bernoulli trials. In each trial, p = probability of hitting the target = $\frac{3}{4}$ and q = probability of not hitting the target = $\frac{1}{4}$.

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$

$$= {}^nC_x \left(\frac{1}{4}\right)^{n-x} \left(\frac{3}{4}\right)^x = {}^nC_x \frac{3^x}{4^n}$$

Now, given that,

P(hitting the target at least once) > 0.99

i.e. $P(x \geq 1) > 0.99$

Therefore, $1 - P(x = 0) > 0.99$

$$\Rightarrow 1 - {}^nC_0 \frac{1}{4^n} > 0.99 \Rightarrow {}^nC_0 \frac{1}{4^n} < 0.01$$

$$\Rightarrow \frac{1}{4^n} < 0.01 \Rightarrow 4^n > \frac{1}{0.01} = 100$$

The minimum value of n to satisfy the inequality (1) is 4.

Thus, the shooter must fire 4 times.

40. A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Ans:

Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $P(S) = \frac{1}{6}$, $P(F) = \frac{5}{6}$

P(A wins in the first throw) = P(S) = $\frac{1}{6}$

A gets the third throw, when the first throw by A and second throw by B result into failures.

Therefore, P(A wins in the 3rd throw) = P(FFS) = P(F)P(F)P(S) = $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \left(\frac{5}{6}\right)^2 \times\frac{1}{6}$

P(A wins in the 5th throw) = P (FFFFS) = $\left(\frac{5}{6}\right)^4 \times\frac{1}{6}$ and so on.

Hence, P(A wins) = $\frac{1}{6} + \left(\frac{5}{6}\right)^2 \times\frac{1}{6} + \left(\frac{5}{6}\right)^4 \times\frac{1}{6} + ... = \frac{\frac{1}{6}}{1-\frac{25}{36}} = \frac{6}{11}$

P(B wins) = 1 – P (A wins) = $1-\frac{6}{11}=\frac{5}{11}$

41. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

Ans:

Let E_1 : the event that selected person is a male and E_2 : the event that selected person is a female.

E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$P(E_1) = P(E_2) = \frac{1}{2}$

Let E : the event that selected person is grey haired.

Then $P(E/E_1) = \frac{5}{100}=\frac{1}{20}$ and $P(E/E_2) = \frac{0.25}{100}=\frac{1}{400}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}=\frac{\frac{1}{2}\times\frac{1}{20}}{\frac{1}{2}\times\frac{1}{20}+\frac{1}{2}\times\frac{1}{400}}=\frac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{400}}=\frac{\frac{1}{20}}{\frac{21}{400}}=\frac{20}{21}$$

42. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

Ans:

A person can be either right handed or left handed. It is given that 90% of the people are right handed.

Therefore $p = \frac{90}{100}=\frac{9}{10}$

and $q = 1-p=1-\frac{9}{10}=\frac{1}{10}$, $n = 10$

Clearly, X has a binomial distribution with $n = 10$, $p = \frac{9}{10}$ and $q = \frac{1}{10}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$

$$P(X=r)={}^{10}C_r\left(\frac{9}{10}\right)^r\left(\frac{1}{10}\right)^{10-r}$$

Required probability, $P(X \le 6)$

$= 1 - P$ {more than 6 are right handed $(7 \le X \le 10)$]

$$=1-\sum_{r=7}^{10}{}^{10}C_r\left(\frac{9}{10}\right)^r\left(\frac{1}{10}\right)^{10-r}$$

43. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that (i) all will bear 'X' mark. (ii) not more than 2 will bear 'Y' mark. (iii) at least one ball will bear 'Y' mark. (iv) the number of balls with 'X' mark and 'Y' mark will be equal.

Ans:

It is case of Bernoulli trials with $n = 6$. Let success be defined as drawing a ball marked X.

$p = P$ (a success in a single draw) = $\frac{10}{25}=\frac{2}{5}$

and $q = 1-p=1-\frac{2}{5}=\frac{3}{5}$

Clearly, Z has a binomial distribution with $n = 6$, $p = \frac{2}{5}$ and $q = \frac{3}{5}$

Therefore $P(Z = r) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$

$$P(Z = r) = {}^6C_r \left(\frac{2}{5}\right)^r \left(\frac{3}{5}\right)^{6-r}$$

(i) P (all bear mark X) = P (6 success) = ${}^6C_6 p^6 q^0 = \left(\frac{2}{5}\right)^6$

(ii) P (not more than 2 bear mark Y)
= P (not less than 4 bear mark X) = P (atleast 4 successes)
= $P(4) + P(5) + P(6) = {}^6C_4 p^4 q^2 + {}^6C_5 p^5 q^1 + {}^6C_6 p^6 q^0$

$$= 15\left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^2 + 6\left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right)^1 + \left(\frac{2}{5}\right)^6$$

$$= \left(\frac{2}{5}\right)^4 \left[\frac{27}{5} + \frac{36}{25} + \frac{4}{25}\right] = \left(\frac{2}{5}\right)^4 \left[\frac{135+36+4}{25}\right]$$

$$= \left(\frac{2}{5}\right)^4 \times \frac{175}{25} = 7\left(\frac{2}{5}\right)^4$$

(iii) P (atleast one ball will bear mark Y) = P (atleast one failure)

= P (atmost five successes) = $1 - P(6) = 1 - \left(\frac{2}{5}\right)^6$

(iv) Required probability = P (three successes and three failures)

$$= P(3) = {}^6C_3 p^3 q^3 = 20\left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^3 = 20 \times \frac{8}{125} \times \frac{27}{125} = \frac{864}{3125}$$

44. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Ans:
It is a case of Bernoulli trials, where success is crossing a hurdle successfully without knocking it down and $n = 10$.

$p = P$ (success) $= \frac{5}{6}$

$q = 1 - p = 1 - \frac{5}{6} = \frac{1}{6}$

Let X be the random variable that represents the number of times the player will knock down the hurdle.

Clearly, X has a binomial distribution with $n = 10$ and $p = \frac{5}{6}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$

$$P(X = r) = {}^{10}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{10-r}$$

P (player knocking down less than 2 hurdles) = $P(x < 2)$
= $P(0) + P(1) = {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9$

$$= \left(\frac{5}{6}\right)^{10} + 10\left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9 = \left(\frac{5}{6}\right)^9 \left[\frac{5}{6} + \frac{10}{6}\right] = \left(\frac{5}{6}\right)^9 \times \frac{15}{6} = \frac{5}{2} \times \left(\frac{5}{6}\right)^9 = \frac{5^{10}}{2 \times 6^9}$$

45. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Ans:

When a die is rolled once, probability of obtaining a six is $\frac{1}{6}$ and that of not obtaining a six is

$1-\frac{1}{6}=\frac{5}{6}$

Let p = $\frac{1}{6}$ and q = $\frac{5}{6}$

Clearly, X has a binomial distribution.

Therefore P(X = x) = ${}^nC_x q^{n-x} p^x$, x = 0, 1, 2,...,n

$$P(X=r)={}^5C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{5-r}$$

$\therefore$ P (obtaining third six in the sixth throw)

= P(obtaining two sixes in first five throws and a six in the sixth throw)

= P (obtaining two sixes in first five throws) × $\frac{1}{6}$

$$= {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3\times\frac{1}{6}=10\times\frac{1}{36}\times\frac{125}{216}=\frac{625}{23328}$$

46. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Ans:

Let E_1 : the event that the patient follows meditation and yoga and E_2 : the event that the patient uses drug.

Therefore E_1 and E_2 are mutually exclusive events and

$P(E_1) = P(E_2) = \frac{1}{2}$

$\therefore$ $P(E/E_1) = \frac{40}{100}\left(1-\frac{30}{100}\right)=\frac{28}{100}$

$P(E/E_2) = \frac{40}{100}\left(1-\frac{25}{100}\right)=\frac{30}{100}$

Let E : the event that the selected patient suffers a heart attack

By using Baye's theorem, we obtain

P (patient who suffers heart attack follows meditation and yoga) =

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{28}{100}}{\frac{1}{2}\times\frac{30}{100}+\frac{1}{2}\times\frac{28}{100}}=\frac{\frac{14}{100}}{\frac{15}{100}+\frac{14}{100}}=\frac{14}{15+14}=\frac{14}{29}$$

47. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Ans:

Let E_1 : red ball is transferred from bag I to bag II

and E_2 : black ball is transferred from bag I to bag II

$\therefore$ E_1 and E_2 are mutually exclusive and exhaustive events.

$$P(E_1) = \frac{3}{3+4} = \frac{3}{7}$$

$$P(E_2) = \frac{4}{3+4} = \frac{4}{7}$$

$$\therefore P(E/E_1) = \frac{4+1}{(4+1)+5} = \frac{5}{10} = \frac{1}{2}$$

$$P(E/E_2) = \frac{4}{4+(5+1)} = \frac{4}{10} = \frac{2}{5}$$

Let E be the event that the ball drawn is red. When a red ball is transferred from bag I to II.

When a black ball is transferred from bag I to II.

By using Baye's theorem, we obtain

$$P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$$

$$= \frac{\frac{4}{7}\times\frac{2}{5}}{\frac{3}{7}\times\frac{1}{2}+\frac{4}{7}\times\frac{2}{5}} = \frac{\frac{8}{35}}{\frac{3}{14}+\frac{8}{35}} = \frac{\frac{8}{35}}{\frac{305+112}{14\times 35}} = \frac{8\times 14}{217} = \frac{16}{31}$$

....