CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 06 marks

NCERT Important Questions & Answers

1. **Determine whether each of the following relations are reflexive, symmetric and transitive : (iv) Relation R in the set Z of all integers defined as $R = \{(x, y): x - y$ is an integer $\}$**
   Ans:
   For reflexive put $y = x$, $x - x = 0$ which is an integer for all $x \in Z$. So, R is reflexive on Z.
   For symmetry let $(x,y) \in R$, then $(x - y)$ is an integer λ and also $y - x = -\lambda$ [$\therefore\ \lambda \in Z \Rightarrow -\lambda \in Z$]
   $\therefore\ y - x$ is an integer $\Rightarrow (y, x) \in R$. So, R is symmetric.
   For transitivity let $(x,y) \in R$ and $(y, z) \in R \Rightarrow x - y$ = integer and $y - z$ = integers, then $x - z$ is also an integer
   $\therefore (x, z) \in R$. So, R is transitive.

2. **Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.**
   Ans:
   We have $R = \{(a,b) : a \le b^2\}$, where $a, b \in R$
   For reflexivity, we observe that $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true.
   So, R is not reflexive as $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$
   For symmetry, we observe that $-1 \le 3^2$ but $3 > (-1)^2$
   $\therefore (-1, 3) \in R$ but $(3, -1) \notin R$.
   So, R is not symmetric.
   For transitivity, we observe that$2 \le (-3)^2$ and $-3 \le (1)^2$ but $2 > (1)^2$
   $\therefore (2, -3) \in R$ and $(-3,1) \in R$ but $(2, 1) \notin R$. So, R is not transitive.
   Hence, R is neither reflexive, nor symmetric and nor transitive.

3. **Show that the relation R in R defined as $R = \{a, b) : a \le b\}$, is reflexive and transitive but not symmetric.**
   Ans:
   We have $R = \{(a,b) : a \le b\}$. Let $a,b \in R$.
   Reflexive: for any $a \in R$ we have $a \le a$. So, R is reflexive.
   Symmetric: we observe that $(2, 3) \in R$ but $(3, 2) \notin R$. So, R is not symmetric.
   Transitivity: $(a,b) \in R$ and $(b, c) \in R \Rightarrow a \le b$ and $b \le c \Rightarrow a \le c \Rightarrow (a,c) \in R$
   So, R is transitive. Hence, R is reflexive and transitive but not symmetric.

4. **Check whether the relation R in R defined by $R = \{(a, b): a \le b^3\}$ is reflexive, symmetric or transitive.**
   Ans:
   Given that $R = \{(a, b): a \le b^3\}$
   It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \in R$ as $\frac{1}{2} < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
   So, R is not reflexive.
   Now, (1, 2) (as $1 < 2^3 = 8$)
   But $(2, 1) \notin$ R (as $2^3 > 1$)
   So, R is not symmetric.

We have $\left(3,\frac{3}{2}\right),\left(\frac{3}{2},\frac{6}{5}\right)\in R$ as $3>\left(\frac{3}{2}\right)^3$ *and* $\frac{3}{2}<\left(\frac{6}{5}\right)^3$

But $\left(3,\frac{6}{5}\right)\notin R$ *as* $3>\left(\frac{6}{5}\right)^3$

Therefore, *R* is not transitive.
Hence, *R* is neither reflexive nor symmetric nor transitive.

5. Show that the relation R in the set *A* = {1, 2, 3, 4, 5} given by R = {(a, b) : |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of 2, 4}.

Ans:

Given that A = {1, 2, 3, 4, 5} and R = {(a, b) : |a – b| is even}
It is clear that for any element *a* ∈*A*, we have (which is even).
∴R is reflexive.
Let (*a*, *b*) ∈ R.
⇒ |a – b| is even
⇒(a – b) is even
⇒– (a – b) is even
⇒(b – a) is even
⇒ |b – a| is even
⇒(*b*, *a*) ∈ R
∴R is symmetric.
Now, let (*a*, *b*) ∈ R and (*b*, *c*) ∈ R.
⇒ |a – b| is even and |b – c| is even
⇒ (a – b) is even and (b – c) is even
⇒ (a – c) = (a – b) + (b – c) is even (Since, sum of two even integers is even)
⇒ |a – c| is even
⇒ (*a*, *c*) ∈ R
∴R is transitive.
Hence, R is an equivalence relation.
Now, all elements of the set {1, 2, 3} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.
Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.
Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

6. Show that each of the relation R in the set $A=\{x\in Z:0\le x\le 12\}$, given by R = {(a, b): |a – b| is a multiple of 4} is an equivalence relation. Find the set of all elements related to 1.

Ans:

$A=\{x\in Z:0\le x\le 12\}=\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and
R = {(a, b): |a – b| is a multiple of 4}
For any element *a* ∈A, we have (*a*, *a*) ∈ R ⇒ |a – a| = 0 is a multiple of 4.
∴R is reflexive.
Now, let (*a*, *b*) ∈ R ⇒|a – b| is a multiple of 4.
⇒|–(a – b)| is a multiple of 4
⇒|b – a| is a multiple of 4.
⇒ (*b*, *a*) ∈ R

∴ R is symmetric.
Now, let $(a, b), (b, c) \in$ R.
⇒|a – b| is a multiple of 4 and |b – c| is a multiple of 4.
⇒(a – b) is a multiple of 4 and (b – c) is a multiple of 4.
⇒(a – b + b – c) is a multiple of 4
⇒(a – c) is a multiple of 4
⇒|a – c| is a multiple of 4
⇒ $(a, c) \in$ R
∴ R is transitive.
Hence, R is an equivalence relation.
The set of elements related to 1 is {1, 5, 9} since
|1 – 1| = 0 is a multiple of 4
|5 – 1| = 4 is a multiple of 4
|9 – 1| = 8 is a multiple of 4

7. In each of the following cases, state whether the functions is one-one, onto or bijective. Justify answer.
(i) $f : R \to R$ defined by $f(x) = 3 - 4x$
(ii) $f : R \to R$ defined by $f(x) = 1 + x^2$
Ans:
(i) Here, $f : R \to R$ is defined by $f(x) = 3 - 4x$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
⇒ $3 - 4x_1 = 3 - 4x_2$
⇒ $-4x_1 = -4x_2$
⇒ $x_1 = x_2$
Therefore, f is one-one.
For any real number y in R, there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$
Therefore, f is onto. Hence, f is bijective.

(ii) Here $f : R \to R$ is defined as $f(x) = 1 + x^2$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
⇒ $1 + x_1^2 = 1 + x_2^2$
⇒ $x_1^2 = x_2^2$
⇒ $x_1 = \pm x_2$
For instance, $f(1) = f(-1) = 2$
Therefore, $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$
Therefore, f is not one-one.
Consider an element − 2 in co-domain R.
It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.
Thus, there does not exist any x in domain R such that $f(x) = -2$.
Therefore, f is not onto. Hence, f is neither one-one nor onto.

8. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$ is f one-one and onto ? Justify your answer.
Ans:
Here, $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$ is defined as $f(x) = \left(\frac{x-2}{x-3}\right)$
Let $x, y \in A$ such that $f(x) = f(y)$

$$\Rightarrow \frac{x-2}{x-3}=\frac{y-2}{y-3} \Rightarrow (x-2)(y-3)=(y-2)(x-3)$$
$$\Rightarrow xy-3x-2y+6=xy-3y-2x+6$$
$$\Rightarrow -3x-2y=-3y-2x$$
$$\Rightarrow 3x-2x=3y-2y \Rightarrow x=y$$

Therefore, f is one- one. Let $y \in B = R - \{1\}$. Then, $y \neq 1$
The function f is onto if there exists $x \in A$ such that $f(x) = y$.
Now, $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3}=y \Rightarrow x-2=xy-3y$$
$$\Rightarrow x(1-y)=-3y+2$$
$$\Rightarrow x=\frac{2-3y}{1-y} \in A \ \ [y \neq 1]$$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y$$

Therefore, f is onto. Hence, function f is one-one and onto.

9. If $f(x)=\frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, show that (fof)(x) = x, for all $x \neq \frac{2}{3}$. What is the inverse of f?

Ans:

Given that $f(x)=\frac{4x+3}{6x-4}, x \neq \frac{2}{3}$

Then $(fof)(x)= f(f(x))= f\left(\frac{4x+3}{6x-4}\right)$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

Therefore (fof)(x) = x, for all $x \neq \frac{2}{3}$

Hence, the given function f is invertible and the inverse of f is itself.

10. Show that $f:[-1,1] \rightarrow R$, given by $f(x)=\frac{x}{x+2}, x \neq -2$, is one-one. Find the inverse of the function $f:[-1,1] \rightarrow$ Range f.

Ans:

Given that $f:[-1,1] \rightarrow R$, given by $f(x)=\frac{x}{x+2}, x \neq -2$,

Let f(x) = f(y)

$$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2} \Rightarrow xy+2x=xy+2y$$
$$\Rightarrow 2x=2y \Rightarrow x=y$$

Therefore, f is a one-one function.

Let $y=\frac{x}{x+2}\Rightarrow x=xy+2y\Rightarrow x=\frac{2y}{1-y}$

So, for every *y* except 1 in the range there exists *x* in the domain such that *f*(*x*) = *y*. Hence, function *f* is onto.

Therefore, $f:[-1,1]\rightarrow$ Range *f* is one-one and onto and therefore, the inverse of the function $f:[-1,1]\rightarrow$ Range *f* exists.

Let *y* be an arbitrary element of range *f*.

Since, $f:[-1,1]\rightarrow$ Range *f* is onto, we have *y* = *f*(*x*) for some $x\in[-1,1]$

$\Rightarrow y=\frac{x}{x+2}\Rightarrow x=xy+2y\Rightarrow x=\frac{2y}{1-y}, y\neq 1$

Now, let us define *g*: Range $f\rightarrow[-1,1]$ as $g(y)=\frac{2y}{1-y}, y\neq 1$

$$(gof)(x)=g(f(x))=g\left(\frac{x}{x+2}\right)=\frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)}=\frac{2x}{x+2-x}=\frac{2x}{2}=x$$

and $(fog)(y)=f(g(y))=f\left(\frac{2y}{1-y}\right)=\frac{\left(\frac{2y}{1-y}\right)}{\left(\frac{2y}{1-y}\right)+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=y$

Therefore, *gof* = *fog* = I_R, Therefore, $f^{-1}=g$

Therefore, $f^{-1}(y)=\frac{2y}{1-y}, y\neq 1$

11. Consider *f* : *R*→ *R* given by *f* (*x*) = 4*x* + 3. Show that *f* is invertible. Find the inverse of *f* .

Ans:

Here, $f:R\rightarrow R$ is given by *f*(*x*) = 4*x* + 3

Let $x,y\in R$, such that *f*(*x*) = *f*(*y*)

$\Rightarrow 4x+3=4y+3$

$\Rightarrow 4x=4y\Rightarrow x=y$

Therefore, *f* is a one-one function.

Let *y* = 4*x* + 3

$\Rightarrow$ There exist, $x=\left(\frac{y-3}{4}\right)\in R, \forall y\in R$

Therefore, for any $y\in R$, there exist $x=\left(\frac{y-3}{4}\right)\in R$ such that

$f(x)=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y$

Therefore, *f* is onto function.

Thus, *f* is one-one and onto and therefore, f^{-1} exists.

Let us define $g:R\rightarrow R$ by $g(x)=\frac{x-3}{4}$

Now, $(gof)(x)=g(f(x))=g(4x+3)=\frac{(4x+3)-3}{4}=x$

and $(fog)(y)=f(g(y))=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y$

Therefore, *gof* = *fog* = I_R

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \frac{y-3}{4}$

12. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given by $f^{-1}y = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Ans: Here, function $f : R_+ \to [4,\infty]$ is given as $f(x) = x^2 + 4$

Let $x, y \in R_+$, such that $f(x) = f(y)$

$\Rightarrow x^2 + 4 = y^2 + 4 \Rightarrow x^2 = y^2$

$\Rightarrow x = y$ [as $x = y \in R_+$]

Therefore, f is a one-one function.

For $y \in [4,\infty)$, let $y = x^2 + 4$

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in R_+$, There exists $x = \sqrt{y-4} \in R_+$ such that

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$

Therefore, f is onto. Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g : [4,\infty) \to R_+$ by $g(y) = \sqrt{y-4}$

Now, $gof(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$

and $fog(y) = f(g(y)) = f(\sqrt{y-4}) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$

Therefore, $gof = I_{R_+}$ *and* $fog = I_{[4,\infty)}$

Hence, f is invertible and the inverse of f if given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

13. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$.

Ans:

Here, function $f : R_+ \to [-5,\infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be any arbitrary element of $[-5,\infty)$.

Let $y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x + 1)^2 - 1 - 5 = (3x + 1)^2 - 6$

$\Rightarrow (3x + 1)^2 = y + 6$

$\Rightarrow (3x + 1)^2 = \sqrt{y+6}$ [as $y \geq -5 \Rightarrow y + 6 \geq 0$]

$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

Therefore, f is onto, thereby range $f = [-5,\infty)$.

Let us define $g : [-5,\infty) \to R_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$

Now, $(gof)(x) = g(f(x)) = g(9x^2 + 6x - 5) = g((3x + 1)^2 - 6)$

$= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3} = \frac{3x+1-1}{3} = x$

and $(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right) = \left[3\left(\frac{\sqrt{y+6}-1}{3}\right) + 1\right]^2 - 6$

$= \left(\sqrt{y+6}\right)^2 - 6 = y + 6 - 6 = y$

Therefore, $gof = I_{R_+}$ *and* $fog = I_{[-5,\infty)}$

Hence, *f* is invertible and the inverse of *f* if given by $f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}$

14. Let * be the binary operation on *N* given by *a* * *b* = LCM of *a* and *b*.
(i) Find 5*7 , 20*16 (ii) Is * commutative?
(iii) Is * associative? (iv) Find the identity of * in *N*
(v) Which elements of *N* are invertible for the operation *?
Ans:
The binary operation on *N* is defined as *a***b* = LCM of *a* and *b*.
(i) We have 5 *7 = LCM of 5 and 7 = 35 and 20*16 = LCM of 20 and 16 = 80

(ii) It is known that LCM of *a* and *b* = LCM of *b* and *a* for $a, b \in N$.
Therefore, *a***b* = *b***a*. Thus, the operation * is commutative.

(iii) For $a, b, c \in N$, we have
(*a***b*) **c* = (LCM of *a* and *b*) **c* = LCM of *a*, *b*, and *c*
*a** (*b***c*) = *a* * (LCM of *b* and *c*) = LCM of *a*, *b*, and *c*
Therefore, (*a***b*) **c* = *a** (*b***c*). Thus, the operation is associative.

(iv) It is known that LCM of *a* and 1 = *a* = LCM of 1 and *a*, $a \in N$.
*a**1 = *a* = 1**a*, $a \in N$
Thus, 1 is the identity of * in *N*.

(v) An element *a* in *N* is invertible with respect to the operation *, if there exists an element *b* in *N* such that *a***b* = *e* = *b***a*.
Here, *e* = 1. This means that
LCM of *a* and *b* = 1 = LCM of *b* and *a*
This case is possible only when *a* and *b* are equal to 1.
Thus, 1 is the only invertible element of *N* with respect to the operation *.

15. Let * be the binary operation on *N* defined by *a*b* =HCF of *a* and *b*. Is * commutative? Is * associative? Does there exist identity for this binary operation on *N*?**
Ans:
The binary operation * on *N* is defined as *a***b* = HCF of *a* and *b*.
It is known that HCF of *a* and *b* = HCF of *b* and *a* for $a,b \in N$.
Therefore, *a***b* = *b***a*. Thus, the operation is commutative.
For $a,b,c \in N$, we have (*a***b*)**c* = (HCF of *a* and *b*)**c* = HCF of *a*,*b* and *c*
*a**(*b***c*) = *a**(HCF of *b* and *c*) = HCF of *a*,*b*, and *c*
Therefore, (*a***b*)**c* = *a**(*b***c*)
Thus, the operation * is associative.
Now, an element $e \in N$ will be the identity for the operation if *a***e* = *a* = *e***a*, $\forall\ a \in N$.
But this relation is not true for any $a \in N$.
Thus, the operation * does not have identity in *N*.

16. Let * be a binary operation on the set *Q* of rational number as follows :
(i) *a*b* = *a* − *b* (ii) *a***b* = $a^2 + b^2$ (iii) *a***b* = *a* + *ab***
(iv) *a*b* = $(a-b)^2$ (v) *a***b* = $\frac{ab}{4}$ (vi) *a***b* = ab^2**
Find which of the binary operation are commutative and which are associative?
Ans:
(i) On *Q*, the operation * is defined as *a***b* = *a* − *b*.
It can be observed that for $2,3,4 \in Q$, we have 2*3 = 2 − 3 = −1 and 3*2 = 3 − 2 = 1
$\Rightarrow 2*3 \neq 3*2$
Thus, the operation is not commutative.

It can also be observed that
$(2*3)*4 = (-1)*4 = -1 - 4 = -5$
and $2*(3*4) = 2*(-1) = 2 - (-1) = 3$
$2*(3*4) \neq 2*(3*4)$
Thus, the operation * is not associative.

(ii) On Q, the operation * is defined as $a*b = a^2 + b^2$.
For $a,b \in Q$, we have
$a*b = a^2 + b^2 = b^2 + a^2 = b*a$
Therefore, $a*b = b*a$
Thus, the operation * is commutative.
It can be observed that
$(1*2)*3 = (1^2 + 2^2)*3\ (1 + 4)*4 = 5*4 = 5^2 + 4^2 = 41$
and $1*(2*3) = 1*(2^2 + 3^2) = 1*(4 + 9) = 1*13 = 1^2 + 13^2 = 170$
$\Rightarrow (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in Q$
Thus, the operation * is not associative.

(iii) On Q, the operation is defined as $a*b = a + ab$
It can be observed that
$1*2 = 1 + 1 \times 2 = 1 + 2 = 3$, $2*1 = 2 + 2 \times 1 = 2 + 2 = 4$
$\Rightarrow 1*2 \neq 2*1$ where $1, 2 \in Q$
Thus, the operation * is not commutative.
It can also be observed that
$(1*2)*3 = (1 + 1 \times 2)*3 = 3*3 = 3 + 3 \times 3 = 3 + 9 = 12$
and $1*(2*3) = 1*(2 + 2 \times 3) = 1*8 = 1 + 1 \times 8 = 9$
$\Rightarrow (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in Q$
Thus, the operation * is not associative.

(iv) On Q, the operation * is defined by $a*b = (a - b)^2$.
For $a,b \in Q$, we have $a*b = (a - b)^2$ and $b*a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$
Therefore, $a*b = b*a$
Thus, the operation * is commutative. It can be observed that
$(1*2)*3 = (1 - 2)^2 * 3 = (\ 1) * 3 = (1 - 3)^2 = 4$
and $1*(2*3) = 1*(2 - 3)^2 = 1*(1) = (1 - 1)^2 = 0$
$\Rightarrow (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in Q$
Thus, the operation * is not associative.

(v) On Q, the operation * is defined as $a*b = \frac{ab}{4}$

For $a,b \in Q$, we have $a*b = \frac{ab}{4} = \frac{ba}{4} = b*a$

Therefore, $a*b = b*a$
Thus, the operation * is commutative.

For $a,b,c \in Q$, we have $a*(b*c) = \frac{ab}{4}*c = \frac{\frac{ab}{4}.c}{4} = \frac{abc}{16}$ and $a*(b*c) = a*\frac{bc}{4} = \frac{a.\frac{bc}{4}}{4} = \frac{abc}{16}$
Therefore, $(a*b)*c = a*(b*c)$. Thus, the operation * is associative.

(vi) On Q, the operation is defined as $a*b = ab^2$
It can be observed that for $23 \in Q$
$2*3 = 2 \text{ x } 3^2 = 18$ and $3*2 = 3 \text{ x } 2^2 = 12$
Hence, $2*3 \neq 3*2$
Thus, the operation is not commutative.

It can also be observed that for $1,2,3 \in Q$
$(1*2)*3 = (1.\ 2^2\ )*3 = 4\ *3 = 4.3^2 = 36$
and $1*(2*\ 3) = 1*(2.\ 3^2) = 1\ *18 =1.18^2 = 324$
$\Rightarrow\ (1*2)*3 \neq 1*(2*3)$
Thus, the operation * is not associative.
Hence, the operations defined in parts (ii), (iv), (v) are commutative and the operation defined in part (v) is associative.

17. Show that none of the operation given in the above question has identity.
Ans:
An element $e \in Q$ will be the identity element for the operation if
$a*e = a = e*a,\ a \in Q$
(i) $a*b = a - b$
If $a*e = a, a \neq 0 \Rightarrow a - e = a, a \neq 0 \Rightarrow e = 0$
Also, $e*a = a \Rightarrow e - a = a \Rightarrow e = 2a$
$\Rightarrow\ e = 0 = 2a, a \neq 0$
But the identiry is unique. Hence this operation has no identity.

(ii) $a*b = a^2 + b^2$
If $a*e = a$, then $a^2 + e^2 = \mathrm{a}$
For $a = -2$, $(-2)^2 + \mathrm{e}^2 = 4 + \mathrm{e}^2 \neq -2$
Hence, there is no identity element.

(iii) $a*b = a + ab$
If $a*e = a \Rightarrow a + ae = a \Rightarrow ae = 0 \Rightarrow e = 0, a \neq 0$
Also if $e*a = a \Rightarrow e + ea = a \Rightarrow e = \frac{a}{1-a}, a \neq 1$
$\therefore e = 0 = \frac{a}{1-a}, a \neq 0$
But the identity is unique. Hence this operation has no identify.

(iv) $a*b = (a - b)^2$
If $a*e = a$, then $(a - e)^2 = a$. A square is always positive, so for
$a = -2, (-2 - \mathrm{e})^2 \neq -2$
Hence, there is no identity element.

(v) $a*b = ab/4$
If $a*e = a$, then $ae/4 = a$. Hence, $e = 4$ is the identity element.
$\therefore a*4 = 4*a = 4a/4 = a$

(vi) $a*b = ab^2$
If $a*e = a$ then $ae^2 = a \Rightarrow e^2 = 1 \Rightarrow e = \pm 1$
But identity is unique. Hence this operation has no identity.
Therefore only part (v) has an identity element.

18. Show that the function $f: R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one-one and onto function.
Ans:
It is given that $f: R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$

Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$

It can be observed that if x is positive and y is negative, then we have

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative, then $x > y \Rightarrow x - y > 0$

But, $2xy$ is negative. Then, $2xy \neq x - y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out. Therefore, x and y have to be either positive or negative.

When x and y are both positive, we have $f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$

When x and y are both negative, we have $f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$

Therefore, f is one-one. Now, let $y \in R$ such that $-1 < y < 1$.

If y is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

Therefore, f is onto. Hence, f is one-one and onto.

19. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective.

Ans:

Here, $f : R \to R$ is given as $f(x) = x^3$.

Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow x^3 = y^3$ …(i)

Now, we need to show that $x = y$

Suppose, $x \neq y$, their cubes will also not be equal.

$x^3 \neq y^3$

However, this will be a contradiction to Eq. i).

Therefore, $x = y$. Hence, f is injective.

20. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as $a*b = \begin{cases} a+b, & if a+b<6 \\ a+b-6, & if a-b \geq 6 \end{cases}$. Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $(6 - a)$ being the inverse of a.

Ans:

Let $X = \{0,1,2,3,4,5\}$

The operation * on X is defined as $a*b = \begin{cases} a+b, & if a+b<6 \\ a+b-6, & if a-b \geq 6 \end{cases}$

An element $e \in X$ is the identity element for the operation *, if
$a*e = a = e*a \ \forall a \in X$
For $a \in X$, we observed that
$a*0 = a + 0 = a \quad [\because a \in X \Rightarrow a + 0 < 6]$
$0*a = 0 + a = a \quad [\because a \in X \Rightarrow 0 + a < 6]$
$\therefore a*0 = a = 0*a \ \forall a \in X$

Thus, 0 is the identity element for the given operation *.
An element $a \in X$ is invertible, if there exists $b \in X$ such that $a*b = 0 = b*a$

$$i.e. \begin{cases} a+b=0=b+a, & if a+b<6 \\ a+b-6=0=b+a-6, & if a-b \geq 6 \end{cases}$$

i.e., $a = -b$ or $b = 6 - a$
But $X = \{0,1,2,3,4,5\}$ and $a, b \in X$
Then, $a \neq -b$
Therefore, $b = 6 - a$ is the inverse of a, $a \in X$.
Hence, the inverse of an element $a \in X$, $a \neq 0$ is $(6 - a)$ *i.e.*, $a^{-1} = 6 - a$.

21. Show that the relation R in the set Z of integers given by R = {(*a*, *b*) : 2 divides *a – b*} is an equivalence relation.
Ans:
R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$.
Further, if $(a, b) \in$ R, then 2 divides $a - b$.
Therefore, 2 divides $b - a$.
Hence, $(b, a) \in$ R, which shows that R is symmetric.
Similarly, if $(a, b) \in$ R and $(b, c) \in$ R, then $a - b$ and $b - c$ are divisible by 2.
Now, $a - c = (a - b) + (b - c)$ is even.
So, $(a - c)$ is divisible by 2. This shows that R is transitive.
Thus, R is an equivalence relation in **Z**.

22. Show that if f : A → B and g : B → C are one-one, then gof : A → C is also one-one.
Ans:
Suppose $gof(x_1) = gof(x_2)$
$\Rightarrow g(f(x_1)) = g(f(x_2))$
$\Rightarrow f(x_1) = f(x_2)$, as g is one-one
$\Rightarrow x_1 = x_2$, as f is one-one
Hence, gof is one-one.

23. Determine which of the following binary operations on the set N are associative and which are commutative. (a) $a * b = 1 \ \forall a,b \in N$ (b) $a * b = \dfrac{a+b}{2} \forall a,b \in N$
Ans:
(a) Clearly, by definition $a * b = b * a = 1 \ \forall a,b \in N$.
Also $(a * b) * c = (1 * c) = 1$ and $a * (b * c) = a * (1) = 1, \forall a,b,c \in N$.
Hence R is both associative and commutative.
(b) $a*b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b*a$, shows that * is commutative. Further,

$$(a*b)*c = \left(\frac{a+b}{2}\right)*c = \frac{\left(\frac{a+b}{2}\right)+c}{2} = \frac{a+b+2c}{4}$$

But $a*(b*c) = a*\left(\frac{b+c}{2}\right) = \frac{a+\left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4}$ in general.

Hence, * is not associative.

24. Show that if $f: R-\left\{\frac{7}{5}\right\} \to R-\left\{\frac{3}{5}\right\}$ is defined by $f(x)=\frac{3x+4}{5x-7}$ and $g: R-\left\{\frac{3}{5}\right\} \to R-\left\{\frac{7}{5}\right\}$ is defined by $g(x)=\frac{7x+4}{5x-3}$, then $fog = I_A$ and $gof = I_B$, where, $A = R-\left\{\frac{3}{5}\right\}, B = R-\left\{\frac{7}{5}\right\}$; $I_A(x) = x$, $\forall\ x \in A$, $I_B(x) = x$, $\forall\ x \in B$ are called identity functions on sets A and B, respectively.

Ans:

We have $gof(x) = g\left(\frac{3x+4}{5x-7}\right) = \frac{7\left(\frac{3x+4}{5x-7}\right)+4}{5\left(\frac{3x+4}{5x-7}\right)-3} = \frac{21x+28+20x-28}{15x+20-15x+21} = \frac{41x}{41} = x$

Similarly, $fog(x) = f\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} = \frac{21x+12+20x-12}{35x+20-35+21} = \frac{41x}{41} = x$

Thus, $gof(x) = x$, $\forall\ x \in B$ and $fog(x) = x$, $\forall\ x \in A$, which implies that $gof = I_B$ and $fog = I_A$.

25. Let $f: N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f: N \to S$, where, S is the range of f, is invertible. Find the inverse of f.

Ans:

Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in **N**, which implies that $y = (2x+3)^2 + 6$. This gives $x = \frac{\sqrt{y-6}-3}{2}$, as $y \geq 6$.

Let us define $g: S \to \mathbf{N}$ by $g(y) = \frac{\sqrt{y-6}-3}{2}$

Now $gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x+3)^2 + 6)$

$$= \frac{\sqrt{(2x+3)^2+6-6}-3}{2} = \frac{(2x+3-3)}{2} = x$$

and $fog(y) = f\left(\frac{\sqrt{y-6}-3}{2}\right) = \left(\left(2\left(\frac{\sqrt{y-6}-3}{2}\right)+3\right)^2+6\right)$

$$= \left(\sqrt{y-6}-3+3\right)^2+6 = \left(\sqrt{y-6}\right)^2+6 = y-6+6 = y$$

Hence, $gof = I_{\mathbf{N}}$ and $fog = I_S$. This implies that f is invertible with $f^{-1} = g$.